Chern–Simons Theory and Knot Invariants

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Abstract

Chern–Simons theory is a three dimensional gauge theory which is independent of the metric. The natural observables in Chern–Simons theory, Wilson loops, can be identified with knots in S^3. We present knot theory and knot invariants. We review Chern–Simons theory, then compute Wilson link vacuum expectation values and show that they are invariants in knot theory.

Introduction

In quantum field theory, the most commonly used gauge theory is the Yang-Mills (YM) gauge theory. However, to define YM gauge theory, one has to choose a metric on the manifold. There is an interesting gauge theory that can be defined on a 3-manifold without metric, the Chern–Simons (CS) theory, sometimes also called the Chern–Simons–Witten theory. Since CS theory is defined without a metric, it is known as a *topological* QFT (TQFT). A defining feature of TQFTs is that since there's no metric, there's no inherent time direction, thus they have vanishing Hamiltonian and hence no dynamics. However, they are still interesting objects to study. For example, we will find that CS theory in 3 dimensions leads to invariants of knots. Another reason TQFTs are interesting is that they can be axiomatized, and hence lends to a rigorous study.

In mathematics, knot theory studies *knots* and *links* and is one of the main areas of research of lower dimensional topology. In his landmark paper[10], Witten shows that CS theory is intrinsically related to knot theory. In particular, the vacuum expectation values of links of *Wilson loops*, observable operators in CS theory, are in one to one correspondence with invariants that can be defined on links. We discuss some of his results in this essay.

This essay is organized as follows: In section 1, we follow the development in Baez and Muniain[2] and present the basics of knot theory and introduce the concept of orientation, framing, and skein relations. We also introduce link invariants such as the linking number and the Jones polynomial. In section 2, we review gauge theories, connections, and curvatures and define Chern–Simons theory and define Wilson loops which as basic observables in CS theory. We follow the Witten's paper but also draw ideas from [5] and [9] to briefly discuss the quantization of CS in cases relevant to our analysis. In section 3, we bring knot theory and CS together by computing Wilson loop vacuum expectation values in $U(1)$ and $SU(n)$ theories, and see that they correspond to link invariants.

1 Knot Theory

Before we delve into the relation between gauge theory and knot theory, we have to learn a bit about knots. Intuitively, a knot is a thin piece of string which we tie up somehow and glue the ends together. Two key problems in knot theory we are concerned with are: 1. How do we differentiate two knots? and 2. How do we classify knots?

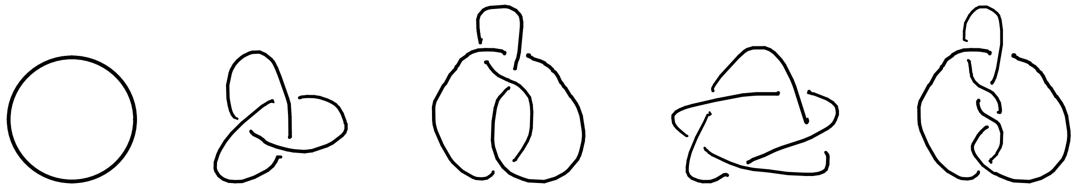
1.1 Knots and Links

Definition 1. A *knot* is an embedding of the circle S^1 in $\mathbb{R}^3$.

We can also put knots together to form a *link*.

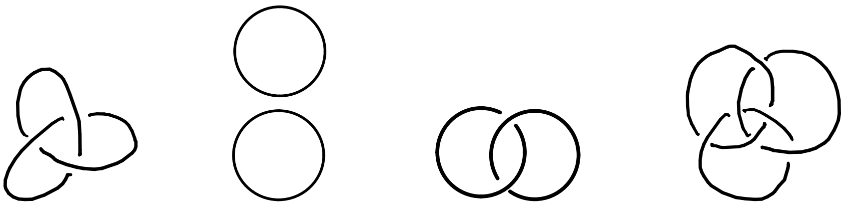
Definition 2. A *link* is an disjoint union of knots. One such knot is called a *component* of the link.

Since a knot is a link, from now on we will only refer to links in general, and to a knot when we want specify a one-component link. Often it's convenient to talk about links in $S^3 = \mathbb{R}^3 \cup \{\infty\}$



(a) *Unknot* **(b)** *Trefoil* 3_1 **(c)** *Figure 8,* 4_1 **(d)** *Cinquefoil* 5_1 **(e)** 5_2

Figure 1: *Some knots. The number indicates the number of crossings and the subscript index is an arbitrary order.*



(a) *Empty link* **(b)** *Knot* **(c)** *Disjoint link* **(d)** *Hopf link* 2_1^2 **(e)** *Borromean rings* 6_2^3

Figure 2: *Some links. The superscript indicates the number of components. Note that if we remove any component of the Borromean rings* **(e)***, the other two components will be unlinked.*

instead of $\mathbb{R}^3$, since we can think of an infinitely extended line as being a knot. For our purposes they are equivalent.

As shown in the previous examples, the standard way to represent links is by a projection onto a two dimensional plane such that there are only a finite number of isolated crossings while keeping track which line is on top and which is on the bottom. This intuitive procedure can be rigorously justified. In particular, one can show that cusps, tangencies, triple crossings, and other singularities can be eliminated by changing the projection by a small amount.

Two link diagrams can look different, yet represent the same link.

Definition 3. Two links L and L' are *ambient isotopic* or simply *isotopic*, denoted $L \sim L'$, if there is a smooth map $\alpha : [0,1] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $\alpha(t, \cdot)$ is a diffeomorphism for every $t \in [0,1]$, $\alpha(0,\cdot)$ is the identity map and $\alpha(1,\cdot)$ maps L to L'. The map α is called an *ambient isotopy.* An equivalence class of ambient isotopic links is called an *isotopy class.*

Intuitively, two links are ambient isotropic if we can deform one into the other smoothly without crossing itself.

From these definitions, we arrive at a natural question. When do two different link diagram represent the same isotopy class? We answer the question with a deceptively simple theorem, which we will not prove.

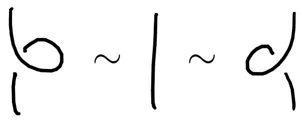
Theorem 4 (Reidemeister)**.** *Two diagrams are ambient isotopic if and only if they are related by a series of* Reidemeister moves.

The Reidemeister moves, shown in Figure 3, are a set three local modifications of the link diagram, usually denoted I, II, and III. We call a local portion of a component of a link a *strand.*

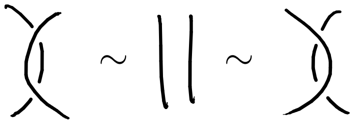
In order to talk about link invariants, we must first equip links with some extra structure. Since a link is a manifold, we can give it an orientation.

Definition 5. A link with an orientation as a manifold is called an *oriented link.*

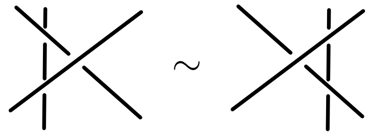
The notion of ambient isotopy carries over to oriented link in the obvious way. Two initially isotopic links can be non isotopic after given an orientation such as the Hopf links in Figure 4.



(a) *The first Reidemeister move (I), twisting and untwisting a strand.*



(b) *The second Reidemeister move (II), a cancelling pair of crossings.*



(c) *The third Reidemeister move (III), sliding a strand underneath a crossing.*

Figure 3: *The three Reidemeister moves.*

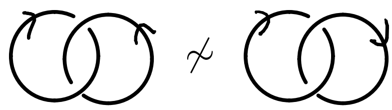


Figure 4: *Two non-isotopic oriented Hopf links.*

The other structure we can give the link is a *framing*.

Definition 6. A *framing* of a link L is a nowhere vanishing vector field V on L such that for every point $p \in L$, $V_p \notin T_pL$, that is, V is nowhere tangent to L. A link with framing is called a *framed link*.

A useful way to visualize framed links is to imagine them as thin ribbons, as in Figure 5 and 6, where the vectors form the width of the ribbon. Note that we don't allow for Möbius strips. Similarly to orientation, we can adapt ambient isotopy to framed links.

Given any link, we can always equip it with a standard framing called the *blackboard framing*, where the vector field V is everywhere orthogonal to the plane of projection of the link diagram, ie. out of the page. We will assume blackboard framing for framed links from now on.

Since we can adapt ambient isotopy to these new structures, we can modify the Reidemeister moves to be compatible with them. The oriented Reidemeister moves are the same as the basic ones, but we must keep track of the orientations. However, the first Reidemeister move fails for framed links, since it introduces an extra twist in the framing. We can modify it by not allowing it to twist or untwist the framed link, like in Figure 7. We call the resulting move I'.

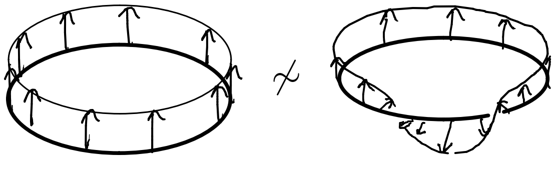


Figure 5: *Two non-isotopic framings of the unknot.*

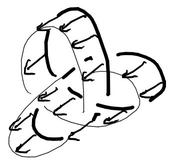


Figure 6: *Trefoil with the blackboard framing.*

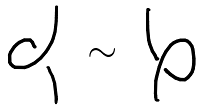


Figure 7: *Modified first Reidemeister move (I') for framed links. 'Flipping over' the twist. Notice the difference in over/under vs. I.*

1.2 Linking Number and Writhe

In order to classify knots and links, we should look for isotopy invariants, ie. quantities calculated from a link diagram invariant under ambient isotopies. There are two simple examples of invariants, the *linking number* and the *writhe.*

Given an oriented link $L = \{K_i\}_{i=1}^r$, where K_i are the component knots, we can classify crossings into two types, positive and negative. The difference is noted in Figure 8.

Theorem 7. *Let $L = \{K_i\}_{i=1}^r$ be an oriented link, C_{ij} the set of crossings between K_i and K_j, $i \neq j$, then the* linking number

$$\mathrm{lk}(K_i, K_j) = \frac{1}{2} \sum_{c \in C_{ij}} \mathrm{sign}(c) \tag{1}$$

is a link invariant.

Proof. As will be the case with other link invariants, we only need to show that the linking number is invariant under the Reidemeister moves. I changes the number of crossings, but on a single knot component, so it doesn't affect the linking number. II cancels out a pair of crossings with opposite signs, it also doesn't affect the linking number. Lastly, III doesn't change the number of crossings or their signs. □

Note that the linking number is not a very powerful invariant. Two links can have the same linking numbers but not be ambient isotopic. For example, the *Whitehead link* in Figure 9 has linking number 0, as does a pair of unlinked unknots. Other more powerful invariants, such as the Jones polynomial, will manage to distinguish these two and show that, for example, the Whitehead link is truly linked.

When the invariant was first discovered by Gauss, it was written in terms of integrals. We will present the integral form here in a differential form following the formalism developed in

(a) *Positive crossing or "overcrossing"* **(b)** *Negative crossing or "undercrossing"*

Figure 8: *Sign of crossings for oriented links.*

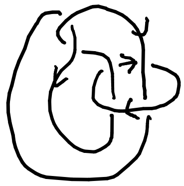


Figure 9: *The Whitehead link, which has 0 linking number but is not isotopic to a 'distant' union of two unknots.*

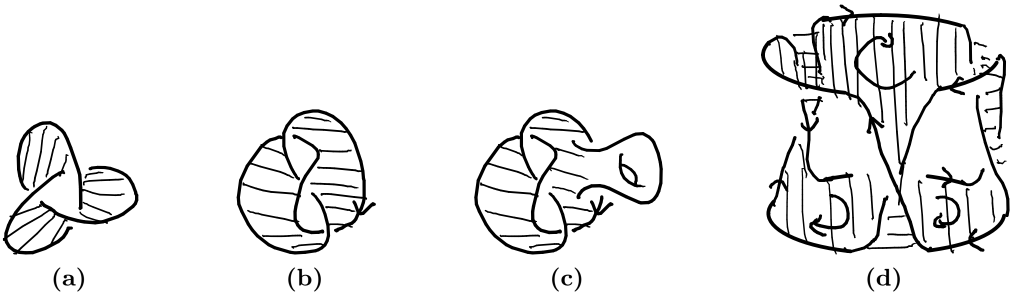


Figure 10: *Some surfaces which have the trefoil as the bounding knot.* ***(a)*** *Möbius strip with three half twists. Not a Seifert surface since it's not orientable.* ***(b)*** *Torus minus a disk, an orientable surface.* ***(c)*** *Same as* ***(b)****, but a surface of higher genus.* ***(d)*** *Seifert surface with induced orientation identified.*

Khesin and Wendt[7] with some modifications. This formalism will be prove to be useful in later sections. First, we cite a surprising result from Seifert[2].

Theorem 8 (Seifert)**.** *Any oriented knot K bounds a compact, connected, and oriented surface Σ such that the orientation on K is induced by the orientation on Σ. The surface Σ is called the* Seifert surface *of K.*

Proposition 9. *Let γ_1, γ_2 be non intersecting curves and choose Σ_1 to be a Seifert surface of γ_1 such that Σ_1 and γ_2 intersect transversally, then the linking number $\mathrm{lk}(\gamma_1, \gamma_2)$ is the number of intersections beween Σ_1 and γ_2, counted with orientations.*

By "counted with orientations" we mean looking at the orientation of Σ_1 and γ_2 at the intersections, comparing it to the orientation of the ambient space, and defining the consistent way to be positive and the other to be negative. It can be shown that $\mathrm{lk}(\gamma_1, \gamma_2)$ is independent of the choice of Σ_1 (as long as it intersects γ_2) and is symmetric in γ_1, γ_2.

We can write the new linking number as

$$\mathrm{lk}(\gamma_1, \gamma_2) = \int_{x\in\Sigma_1} \int_{y\in\gamma_2} \delta^{(2,1)}(x, y) \tag{2}$$

where $\delta^{(2,1)}(x, y)$ is a Dirac delta-type form on $\Sigma_1 \times \gamma_2 \subset \mathbb{R}^3 \times \mathbb{R}^3$, which cancels contributions to the integral from unwanted regions[1]. The indices $(2, 1)$ indicates that $\delta^{(2,1)}$ is a 2-form over x and a 1-form over y. Let's take a brief detour to explore the calculus of these objects a bit more, as they will be useful later. We can integrate out the y component of $\delta^{(2,1)}$ and be left with a δ-type form with support on a closed curve. In other words, consider a 2-form δ_γ with support on a closed oriented curve γ. We want the integral of δ_γ over a surface to pick up the number of times γ intersects the surface counted with orientation. We have

$$\delta_\gamma(x) = \int_{y\in\gamma} \delta^{(2,1)}(x, y). \tag{3}$$

[1]These objects are called *currents*. They can be thought as forms with distribution as coefficients.

Similarly, we can integrate out the x component of $\delta^{(2,1)}$ and be left with a 1-form with support on a surface. choose a Seifert surface Σ of γ, we can define a 1-form supported on Σ as

$$\delta_\Sigma(y) = \int_{x\in\Sigma} \delta^{(2,1)}(x,y). \tag{4}$$

One should be careful in keeping track that δ_γ is a 2-form which can be integrated over a surface, and δ_Σ is a 1-form which can be integrated over a curve. Since these count the intersection points, for some other closed curved γ' with Seifert surface Σ' we must have $\int_{\Sigma'} \delta_\gamma = \int_{\gamma'} \delta_\Sigma$. Hence,

$$\int_{\Sigma'} \delta_\gamma = \int_{\gamma'} \delta_\Sigma = \int_{\partial\Sigma'} \delta_\Sigma = \int_{\Sigma'} \mathrm{d}\delta_\Sigma \tag{5}$$

by Stoke's theorem. Since γ' was some arbitrary closed curve, we have $\mathrm{d}\delta_\Sigma = \delta_\gamma$. Hence, by an abuse of notation, we have $\delta_\Sigma = \mathrm{d}^{-1}\,\delta_\gamma$. This is clearly not a unique construction since the choice of Σ is not unique, so this is only well-defined up to a class of Seifert surfaces (the ones which γ bounds). However, the linking number doesn't depend on the choice of Seifert surface so we're in the clear. Now, getting back to the linking number, we can rewrite

$$\mathrm{lk}(\gamma_1,\gamma_2) = \int_{x\in\Sigma_1} \delta_{\gamma_2}(x) = \int_{x\in\partial\Sigma_1=\gamma_1} \mathrm{d}_x^{-1}\,\delta_{\gamma_2}(x) = \int_{x\in\gamma_1}\int_{y\in\gamma_2} \mathrm{d}_x^{-1}\,\delta^{(2,1)}(x,y) \tag{6}$$

where we used Stoke's theorem in the second equation, and d_x is the exterior derivative with respect to x only. In $\mathbb{R}^3$, we can write the last integral as

$$\mathrm{lk}(\gamma_1,\gamma_2) = \int_{x\in\gamma_1}\int_{y\in\gamma_2} \mathrm{d}_x^{-1}\,\delta^{(2,1)}(x,y) = \oint_{\vec{x}\in\gamma_1}\oint_{\vec{y}\in\gamma_2} \frac{1}{4\pi}\frac{(\vec{x}-\vec{y})\cdot \mathrm{d}\vec{x}\times\mathrm{d}\vec{y}}{|\vec{x}-\vec{y}|^3} \tag{7}$$

which was Gauss' original form of the linking number[2]. The last integral looks very different from our previous definition of linking number. We leave the proof that they are equivalent to the appendix A.1. Note, however, the similarities between (7) and the Biot–Savart law in electromagnetism

$$\vec{B}(\vec{x}) = \frac{\mu_0 I}{4\pi}\oint_{\vec{y}\in C} \frac{\mathrm{d}\vec{y}\times(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^3} \tag{8}$$

where I is a constant current running in a wire C. We can interpret the linking number as the integral of the magnetic field around a closed loop. This is how Gauss likely first came upon the linking number[3].

Another important classical link invariant is the *writhe* of a link, where we just sum over all the crossings instead of crossings of different components.

Theorem 10. *Let L be an oriented, framed link. Then, the* writhe

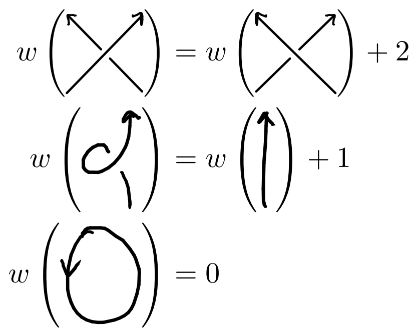
$$w(L) = \sum_{c\in\text{all crossings}} \mathrm{sign}(c) \tag{9}$$

is a link invariant.

We needed the framing here, since the first Reidemeister move changes the writhe while the modified version doesn't. This is because writhe counts the 'twist' in a link, and I introduces new twists/crossings whereas I' maintains the twist and doesn't change the sign of the crossing.

Another way to define writhe is through what is known as *skein relations*. Skein relation will come in handy in defining invariant polynomials in the next subsection. Let's show the invariance of writhe using skein relations.

[2] Not quite. He actually wrote it in terms of ordinary integrals and not closed line integrals.

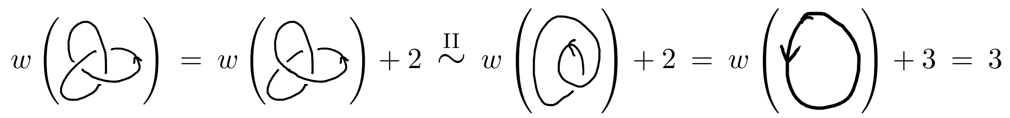


(10)
(11)
(12)

Figure 11: *Skein relations for the writhe.*

Invariance of writhe. II and III don't change the writhe in the same way that they didn't change the linking number. From the skein relation (11), we can see that I' doesn't change the writhe as well since both twists in I' have positive crossings. □

The idea is that we can calculate invariants by repeatedly applying the skein relations which simplifies the diagram until we obtain the unlink. Simpler knot invariants such as the writhe and the Jones polynomial can be calculated by recursively applying skein relations, but the more complicated ones require more complicated algorithms. More precisely, the recursive application of skein relations might not be guaranteed to terminate after some finite number of steps for these invariants[6]. As an example, we compute the writhe of the trefoil in Figure 12.



(13)

Figure 12: *Writhe of the trefoil.*

It turn out the writhe of a link $L = \{K_i\}$ is related to the linking number as

$$w(L) = \sum_{i \neq j} \mathrm{lk}(K_i, K_j) + \sum_i w(K_i) \tag{14}$$

which is why the writhe is sometimes known as the *self-linking number*.

1.3 Invariant Polynomials

There are many more invariants for knots. In fact, Witten showed that there is a link invariant for each finite-dimensional representation of each semi-simple Lie group. In later sections, we will explore two cases in particular, the connection of the linking number and the writhe to representations of $U(1)$, and the connection of the Jones polynomial to the fundamental (Spin-$\frac{1}{2}$) representation of $SU(2)$. In this subsection, we will focus our attention on the construction of the Jones polynomial, and briefly discuss the other invariant polynomials. Note that for these invariants, the exact form of the polynomial and the coefficients don't matter as much as the fact that they don't change under ambient isotopy. Indeed, many of the polynomials are defined inconsistently across the literature (cf. [2], [10]).

Before defining the Jones polynomial, we first construct the Kauffman bracket, which differs

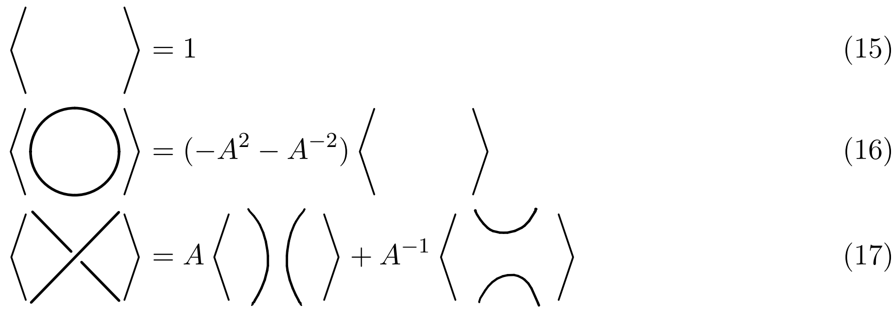


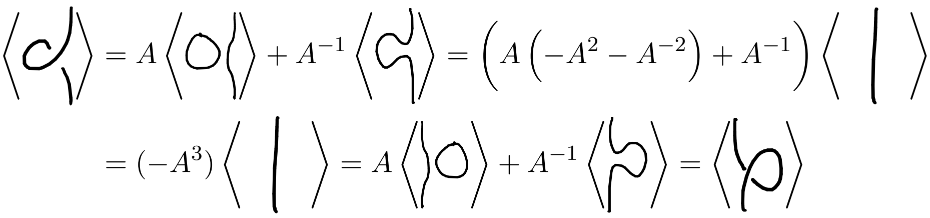
Figure 13: *Skein relations for the Kauffman bracket. The normalization of the empty link is 1.*

from the Jones polynomial by a normalization factor[3]. The Kauffman bracket is an invariant of framed links whereas the Jones polynomial is an invariant for oriented links. We define the bracket of a framed link L, denoted $\langle L \rangle$, by its skein relations (15–17). It is traditionally defined as a function of some variable A. Note that the recursive applications of skein relations is guaranteed to terminate, since the number of crossings is strictly reduced at each step.

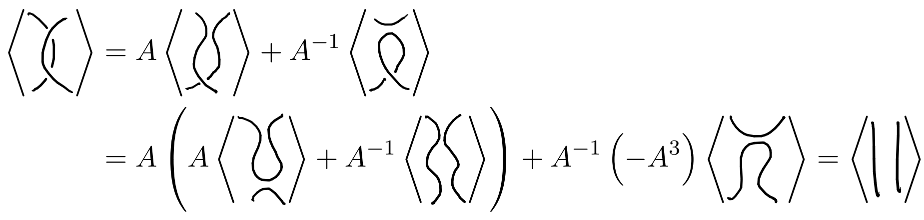
We demonstrate the application of skein relations for the Kauffman bracket by showing that it's invariant under the Reidemeister moves. The Kauffman bracket can be similarly computed for links.

Theorem 11. *Given a framed link L, the Kauffman bracket $\langle L \rangle$ is an invariant.*

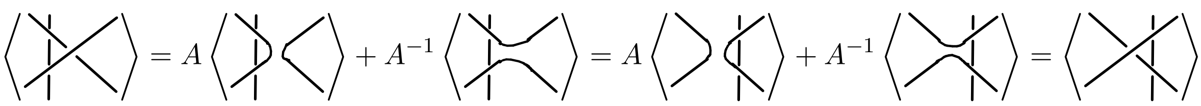
Proof. We show that $\langle L \rangle$ is invariant under I', II, and III.



(a) *Kauffman bracket under I'. Notice in the third equality that untwisting the 'positive' twist multiplies the bracket by $-A^3$.*



(b) *Kauffman bracket under II, using the untwisting factor $-A^3$ in the second equality.*



(c) *Kauffman bracket under III, using II on the second equality.*

Figure 14: *The Kauffman bracket under various Reidemeister moves.*

□

[3]Historically, Jones constructed his polynomial while working on operator algebras in statistical mechanics, but Kauffman's work provides a much simpler way to arrive at the invariant.

From the proof for invariance under I', we see that $\langle L \rangle$ changes by a factor of $-A^3$ under I when we untwist a knot[4]. Thus, it is not an invariant of oriented links. However, we know that the writhe $w(L)$, an invariant for framed, oriented links, changes by $+1$ under the same move. We can combine the two to obtain an invariant on oriented links, the *Jones polynomial.*

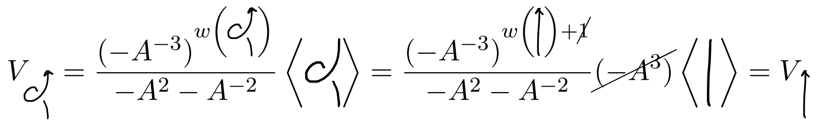
Theorem 12. *Given a oriented link L, the polynomial $V_L(A)$ given by*

$$V_L(A) = \frac{(-A^{-3})^{w(L)}}{-A^2 - A^{-2}} \langle L \rangle \tag{18}$$

is a link invariant. Define $q = A^{-4} \in \mathbb{C}$, then $V_L(q)$ is called the Jones polynomial.

The reason for the factor of $-A^2 - A^{-2}$ in the denominator is because we want to normalize the unknot to have a Jones polynomial of 1.

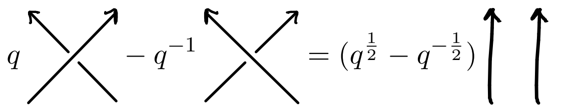
Proof. We only need to show that $V_L(A)$ is invariant under I, since under II and III the invariance follows from that of the Kauffman braket and the writhe. Under I, we have



(19)

□

It's possible to shown from its definition that the Jones polynomial also satisfy a set of skein relations depicted in (20). The unknot is assumed to have a Jones polynomial of 1.



(20)

The Jones polynomial is a stronger invariant than the linking number and the writhe. For example, it can detect that the Whitehead link is truly linked. More importantly, it's able to distinguish some links from their mirror image, where the *mirror image* of a link L is the link with every positive crossing changed to a negative one, and vice versa. By staring at the skein relation for the Jones polynomial (20), it's easy to see that $V_L(q^{-1})$ is the Jones polynomial of the mirror image of the link L.

The Jones polynomial was discovered after the *Alexander–Conway polynomial*, usually denoted ∇_L. Alexander–Conway polynomial is a weaker invariant than the Jones polynomial as it can't distinguish any links from their mirror. There is a two variable generalization of the both of these called the HOMFLY polynomial, named after the initials of its discoverers. It is a even stronger invariant, and able to distinguish some knots where Jones polynomial fails. It is related to the fundamental representation of $SU(n)$ in Chern–Simons theory. There are other link invariants of this type, ie. those related to $SO(n)$ and $Sp(n)$. All of these polynomial obey some skein relations, which we will not reproduce.

[4] $-A^{-3}$ for untwisting a twist with negative crossing.

2 Gauge Theory

In the following sections, let N be an oriented manifold and let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. We consider principal G-bundle E over N. We can cover N with charts and hence obtain local trivializations $\phi_\alpha : E|_{U_\alpha} \to U_\alpha \times G$. The *connection* ∇ on local trivialization ϕ_α has an associated $\mathfrak{g}$-valued 1-form A_α, the *connection form* or the *gauge field*. Within this local trivialization, we have $\nabla = \mathrm{d}+A_\alpha$, so we sometimes refer to A_α as being the connection itself. Moreover, the action of ∇ on a $\mathfrak{g}$-valued p-form ω is

$$\nabla\omega = \mathrm{d}\omega + [A_\alpha, \omega] \tag{21}$$

where $[\omega, \eta] := \omega \wedge \eta - (-1)^{pq}\eta \wedge \omega$ for p-form ω and q-form η is the graded commutator on $\mathfrak{g}$-valued forms. We can also obtain the *curvature 2-form*

$$F_\alpha = \mathrm{d}A_\alpha + A_\alpha \wedge A_\alpha. \tag{22}$$

Given *transitions functions* $g_{\beta\alpha} \in G$ on $U_\alpha \cap U_\beta$, the gauge fields and the curvatures on these patches are related by a *gauge transformation*

$$A_\beta = g_{\beta\alpha} A_\alpha g_{\beta\alpha}^{-1} - \mathrm{d}g_{\beta\alpha} g_{\beta\alpha}^{-1}, \tag{23}$$

$$F_\beta = g_{\beta\alpha} F_\alpha g_{\beta\alpha}^{-1}. \tag{24}$$

From now on, we suppress the indices on the gauge field A and curvature F by restricting ourselves to some local trivialization (or by considering trivial E).

2.1 Chern–Simons Form

The most common and useful gauge theory on N is the Yang-Mills (Maxwell for $U(1)$) theory, which has the following action

$$S_{YM}[A] = \frac{1}{2}\int_N \mathrm{tr}(F \wedge \star F) \tag{25}$$

where $\star$ is the Hodge star. S_{YM} involves the metric because of $\star$. We would like to write down a gauge theory which doesn't involve a metric, ie. one which is *topological*. Since F is a 2-form, if N is $2n$-dimensional, we could try

$$S(A) = \int_N \mathrm{tr}\left(F^{\wedge n}\right) \tag{26}$$

where n^{th} wedge power $\mathrm{tr}\left(F^{\wedge n}\right)$ is called the n^{th} *Chern Form*. We can check that the Chern forms are all closed by the Bianchi identity $\nabla F = 0$. First, we check the Bianchi identity

$$\begin{aligned}\nabla F &= \mathrm{d}F + [A, F] = \mathrm{d}(\mathrm{d}A + A \wedge A) + [A, \mathrm{d}A + A \wedge A] \\ &= \cancel{\mathrm{d}^2 A} + \mathrm{d}A \wedge A - A \wedge \mathrm{d}A + [A, \mathrm{d}A] + \cancel{[A, A \wedge A]} = [\mathrm{d}A, A] + [A, \mathrm{d}A] = 0.\end{aligned} \tag{27}$$

where we used (21). Now, applying the exterior derivative to the k^{th} Chern form,

$$\begin{aligned}\mathrm{d}\,\mathrm{tr}\left(F^{\wedge k}\right) &= \mathrm{tr}\left(\nabla(F \wedge \cdots \wedge F)\right) \\ &= \mathrm{tr}(\nabla F \wedge F^{k-1} + \cdots + F^{k-1} \wedge \nabla F) \\ &= 0.\end{aligned} \tag{28}$$

Let's restrict our attention to N being 4-dimensional, where it will turn out relevant for knot theory. In this case, the second Chern form is actually exact, and we can write down its preimage:

$$\operatorname{tr}(F \wedge F) = \mathrm{d}L_{CS} = \mathrm{d}\operatorname{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right). \tag{29}$$

L_{CS} is called the *Chern–Simons form.* We can view the Chern–Simons form as a boundary term when we integrate the second Chern form over N with boundary. Since L_{CS} is a 3-form that doesn't involve a metric, we can use it as an action of a topological QFT on a 3-dimensional manifold M, with the *Chern–Simons action* S_{CS} given by

$$kS_{CS}[A] = \frac{k}{4\pi}\int_M \operatorname{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right) \tag{30}$$

where k is the coupling constant, sometimes called the *level* of the theory. In the following sections, we will try to consider Chern–Simons action on a general 3-dimensional M whenever possible. However, we will usually restrict our attention to $M = \mathbb{R}^3$ or $M = S^3$, where things are simpler.

Consider the classical Euler-Lagrange equations

$$\begin{aligned} 0 \stackrel{!}{=} \delta S_{CS} &= \delta \frac{1}{4\pi}\int_M \operatorname{tr}\left(A \wedge \mathrm{d}A + \frac{2}{3}A \wedge A \wedge A\right) \\ &= \frac{1}{4\pi}\int_M \operatorname{tr}\left(\partial_j A_k - \partial_k A_j + 2A_j A_k\right)\delta A_i \varepsilon^{ijk}\, \mathrm{d}^3 x \\ &= \frac{1}{2\pi}\int_M \operatorname{tr}\left((\mathrm{d}A + A \wedge A) \wedge \delta A\right) \end{aligned} \tag{31}$$

which vanishes for all δA if the curvature $F = \mathrm{d}A + A \wedge A$ vanishes, ie. if we have a flat connection. Thus, classical Chern–Simons theory looks rather trivial. There are ways to make it interesting, eg., by considering M to have some nontrivial topology. We could also couple the Chern–Simons theory to matter or YM fields, which is usually done in condensed matter theory.

Since we want to consider Chern–Simons as a gauge theory, we should check how it transforms under a gauge transformation. Rewrite (30) as

$$S_{CS}[A] = \frac{1}{4\pi}\int_M \operatorname{tr}\left(A \wedge F - \frac{1}{3}A \wedge A \wedge A\right) \tag{32}$$

and using the cyclic property of the trace as well as gauge transformation laws for A and F (23, 24), we eventually obtain

$$S_{CS}[A^g] - S_{CS}[A] = \frac{1}{4\pi}\int_M \operatorname{tr}\left(\mathrm{d}(g^{-1}\,\mathrm{d}g \wedge A) + \frac{1}{3}(\mathrm{d}g\, g^{-1}) \wedge (\mathrm{d}g\, g^{-1}) \wedge (\mathrm{d}g\, g^{-1})\right) \tag{33}$$

where A^g is the gauge-transformed gauge potential. The first term, which is present in both the abelian and non-abelian theory, integrates to 0 by Stoke's theorem for M without boundary. It turns out that the second term, only present in the non-abelian theory, integrates to $2\pi n$ where n is an integer[7]. This plus the gauge invariance of $e^{ikS_{CS}}$ in the partition function imposes that the coupling constant k must be an integer for non-abelian CS.

2.2 Wilson Loops

Observable quantities in a gauge theory must be gauge invariant. The natural observable in Chern–Simons theory are what's called *Wilson loops*, which arise from integrating A over some closed loop. Before diving into Wilson loops, let's first define the slightly more general concept of a *holonomy.*

Let $\gamma : [0,1] \to M$ be a piecewise smooth path from p to q. The parallel transport map along γ with connection ∇ is a linear map

$$\begin{aligned} H(\gamma, \nabla) : E_p &\to E_q \\ X_p &\mapsto H(\gamma, \nabla) X_p \end{aligned} \tag{34}$$

where $X_p \in E_p$ is a an element of the fibre E_p based at p. Products of these operators can be defined in the usual way when paths are composable. We are interested in the case when γ is a loop, that is, when $p = q$.

Definition 13. Given a loop γ, $H(\gamma, \nabla)$ is call a *holonomy* of ∇ along γ.

Holonomies are basepoint dependent up to conjugation by parallel transport maps between the basepoints. Using the parallel transport equation $\nabla_{\dot{\gamma}} X(t) = 0$ or equivalently $\frac{\mathrm{d}X(t)}{\mathrm{d}t} + A(\dot{\gamma}(t))X(t) = 0$ for $X(t) \in E_{\gamma(t)}$ and appropriately applying gauge transformations, we can check that holonomies transform under gauge transformations as

$$H(\gamma, \nabla^g) = g(p) H(\gamma, \nabla) g(p)^{-1} \tag{35}$$

where $\nabla^g_Y s = g \nabla_Y (g^{-1} s)$ is the gauge-transformed connection for some section s of E and vector field $Y \in TM$. We can solve the parallel transport equation to obtain an explicit expression for the holonomy. Concretely, we have

$$H(\gamma, \nabla) = \mathcal{P} \exp i \oint_\gamma A \tag{36}$$

where $\mathcal{P}$ is the *path ordering operator*, which orders the exponentiated integrals by their parameters, much like the more familiar time ordering operator $\mathcal{T}$. Now, we can build a gauge invariant and basepoint independent quantity by taking the trace.

Definition 14. A *Wilson loop* $W_\rho(\gamma, \nabla)$ is the trace of the holonomy of ∇ along γ in an representation ρ, ie.

$$W_\rho(\gamma, \nabla) := \mathrm{tr}_\rho(H(\gamma, \nabla)) = \mathrm{tr}_\rho \, \mathcal{P} \exp i \oint_\gamma A. \tag{37}$$

The connection is sometimes omitted and we will just write the Wilson loop as $W_\rho(\gamma)$. Since Wilson loops are basepoint independent, it makes sense to talk about Wilson loops of an oriented knot, which we view as a closed loop.

Example 15 (Aharonov–Bohm effect)**.** To see why Wilson loops could be interesting physical quantities, consider a charged particle moving in an electromagnetic field (ie. we consider $U(1)$ gauge theory). The propagation amplitude between points x_i and x_f is given by

$$\begin{aligned} G(x_i, x_f, t) &= \int_{x_i}^{x_f} \mathcal{D}[x] \, \exp\left[i(S_{\text{free}} + S_{\text{int}})\right] \\ &= \int_{x_i}^{x_f} \mathcal{D}[x] \, \exp\left[i\left(S_{\text{free}} + \int_{x_i}^{x_f} A\right)\right]. \end{aligned} \tag{38}$$

Now consider the experimental setup in Figure 15, where a solenoid with some magnetic flux is placed in a double slit experiment. We assume that the solenoid excludes our charged particle entirely, so that γ_1 and γ_2 are topologically distinct paths, ie. we can't smoothly deform one into the other. The magnetic field is uniformly 0 outside the solenoid, so we expect physics to be

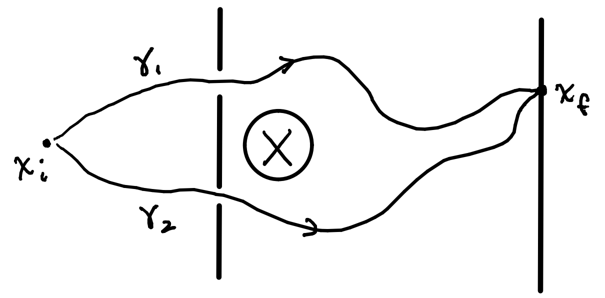


Figure 15: *The Aharanov–Bohm experiment.*

unaffected outside it. However, we find that this is not the case. The total amplitude is the sum of the two path integrals:

$$\begin{aligned} G(x_i, x_f, t) &= \int \mathcal{D}\gamma_1 \exp\left[i\left(S^{(1)}_{\text{free}} + \int_{\gamma_1(t)} A\right)\right] + \int \mathcal{D}\gamma_2 \exp\left[\left(S^{(2)}_{\text{free}} + \int_{\gamma_2(t)} A\right)\right] \\ &= \phi^{(1)}_{\text{free}}(x_f) \exp\left[i \int_{\gamma_1} A\right] + \phi^{(2)}_{\text{free}}(x_f) \exp\left[i\int_{\gamma_2} A\right] \end{aligned} \tag{39}$$

where $\phi^{(i)}_{\text{free}}$ is the phase from the free propagator and $e^{i\int_{\gamma_i} A}$ is constant for trajectories in the same homotopy class by Stoke's theorem. We can take out an overall phase

$$\begin{aligned} G(x_i, x_f, t) &= \phi^{(1)}_{\text{free}}(x_f) \exp\left(i\int_{\gamma_1} A\right) + \phi^{(2)}_{\text{free}}(x_f) \exp\left(i\int_{\gamma_2} A\right) \\ &= \exp\left(i\int_{\gamma_2} A\right)\left(\phi^{(1)}_{\text{free}}(x_f)\exp\left[i\left(\int_{\gamma_1} A - \int_{\gamma_2} A\right)\right] + \phi^{(2)}_{\text{free}}(x_f)\right) \\ &\simeq \phi^{(1)}_{\text{free}}(x_f) \underbrace{\exp\left(i\oint_{\gamma_2^{-1}\circ\gamma_1} A\right)}_{W(\gamma_2^{-1}\circ\gamma_1,\, \mathrm{d}+A)} + \phi^{(2)}_{\text{free}}(x_f) \end{aligned} \tag{40}$$

Hence, there is an extra phase difference between the particles equal to the Wilson loop around the solenoid! This is the *Aharonov–Bohm phase*, which shows that the gauge potential A is a physically measurably object.

In QFT, the physical quantities we will be interested in is the vacuum expectation value (vev) of the Wilson loop $\langle W_\rho(\gamma)\rangle$. The partition function for Chern–Simons theory is given by

$$Z = \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}A] \exp\big(ikS_{CS}[A]\big) \tag{41}$$

where the integral is taken over the space of connections modulo gauge transformations, and $[\mathcal{D}A]$ is the measure in that space. The Wilson loop vev is

$$\langle W_\rho(\gamma)\rangle = Z^{-1}\int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}A] \exp\big(ikS_{CS}[A]\big)\; W_\rho(\gamma). \tag{42}$$

2.3 Quantization of Chern–Simons Theory

We will discuss Witten's technique in quantizing Chern–Simons theory without going too much into detail, as it involves machinery beyond the scope of this essay. In particular, it turns

(a) **(b)**

Figure 16: ***(a)*** *A 3-manifold M cut on a Riemann surface Σ using Heegaard splitting. W_ρ is a Wilson loop in M carrying a representation ρ.* ***(b)*** *Near the cut, M looks like $\mathbb{R} \times \Sigma$. The points p_i and p_j are crossed by the Wilson loop.*

out that quantization reveals that Chern–Simons theory on a 3-manifold M is related to the Wess–Zumino–Witten model of conformal field theory in 2 dimensions, which is an example of the AdS/CFT correspondence[5],[9].

For a general 3-manifold M, the strategy to quantizing Chern–Simons is to cut M into two pieces using what's known as the *Heegaard splitting*, solve the problem on each piece, then glue back together. However, let us instead consider a simpler 3-manifolds of the form $\mathbb{R} \times \Sigma$, where Σ is a 2-dimensional Riemann surface, and we view $\mathbb{R}$ as the time coordinate. Near the cut, which we depict in Figure 16 (b), M does look like this manifold. We can canonically quantize S_{CS} on $\mathbb{R} \times \Sigma$ and obtain a Hilbert space $\mathcal{H}(\Sigma)$ associated to the spacelike slice Σ. The coordinates on $\mathbb{R} \times \Sigma$ are (t, x), so we can write the Chern–Simons action (30) as

$$kS_{CS} = \frac{k}{4\pi} \int_{\mathbb{R}} \mathrm{d}t \int_{\Sigma} \epsilon^{ij} \,\mathrm{tr}\left(A_i \tfrac{\partial}{\partial t} A_j + A_0 F_{ij} \right) \tag{43}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ is the curvature over Σ. As in electromagnetic theory, we can think of A_0 as a Lagrange multiplier which imposes a "Gauss' law" type constraint $F_{ij} = 0$. We can read off the Poisson bracket from the action

$$\{A_i^a(t, x), A_j^b(t, y)\} = \frac{4\pi}{k} \epsilon_{ij} \delta^{ab} \delta^{(2)}(x - y). \tag{44}$$

where $a, b = 1, \ldots, \dim G$ are the Lie algebra indices.

To quantize, we have to both impose the canonical commutation relations as well as the constraints on the system. In theory, the order we do them shouldn't matter and we should get the same quantized system either way. Typically in QFT, we first quantize by changing Poisson bracket into commutators, then impose the constraints of the system for physical wavefunctions. In CS theory, doing things the other way around turns out to be more useful. In other words, we want to first impose the constraint $F_{ij} = 0$ then quantize.

Imposing $F_{ij} = 0$ and identifying the resulting connections related by gauge transformations means we're working in the moduli space $\mathcal{M}$ of flat connections over Σ. It turns out that $\mathcal{M}$ is a *compact* phase space with finite volume with respect to the symplectic volume form. Since there is one quantum state per unit volume of phase space, our Hilbert space $\mathcal{H}(\Sigma)$ will be finite-dimensional.

We restrict our focus to the case where Σ, the shared boundary, is the sphere S^2. If there are Wilson loops running around in M, some of them could cross Σ. We view the crossing points $\{p_i\}_{i=1}^r$ as static charges. We can associate the points $\{p_i\}$ with some representations $\{\rho_i\}_{i=1}^r$. Each ρ_i is associated to a representation space V_{ρ_i}. It turns out that in case of finite

k, the Hilbert space $\mathcal{H}(\Sigma)$ is a subspace of the G-invariant subspace of tensor product of the representation spaces, that is

$$\mathcal{H}(\Sigma, \{p_i\}, \{\rho_i\}) \subset \left(\bigotimes_{i=1}^{r} V_{\rho_i} \right)^G \subset \bigotimes_{i=1}^{r} V_{\rho_i}. \tag{45}$$

where V^G denotes the G-invariant subspace of V.

We let $\{\rho_i\}$ be inherited from the Wilson line. We wish to "orient" the point p_i such that crossing an ingoing line is associated to ρ_i and crossing an outgoing line is associated to its conjugate representation $\bar{\rho}_i$. This orientation is arbitrary but consistent. For example, in Figure 16 (b) we would associate $\bar{\rho}$ to p_i and ρ to p_j. We're interested in the case when $r = 4$, eg. when we have a pair of Wilson lines crossing Σ, and when $\{\rho_i\}$ are the fundamental representation of $SU(N)$. This case will turn out to be important because of the skein relations. The representations are two ρ's and two $\bar{\rho}$'s. We have $V_\rho \otimes V_\rho = V_S \oplus V_A$ where V_S, V_A are the symmetric and antisymmetric representation spaces, each a G-invariant subspace. For example, in the fundamental representation of $SU(2)$ we have $\mathbf{2} \otimes \mathbf{2} = \mathbf{3} \oplus \mathbf{1}$. The G-invariant subspaces on $V_\rho \otimes V_\rho \otimes V_{\bar{\rho}} \otimes V_{\bar{\rho}} = (V_S \oplus V_A) \otimes (\bar{V}_S \oplus \bar{V}_A)$ are then $V_S \otimes \bar{V}_S$ and $V_A \otimes \bar{V}_A$, where the bar indicates the representation space of the conjugate representation. Hence, we have

$$\dim \mathcal{H}(\Sigma) = 2. \tag{46}$$

3 Knot Invariants in Gauge Theory

We need to incorporate Wilson loops into knot theory. The obvious way is taking an oriented knot and view it as a Wilson loop in S^3 with the Chern–Simons action. Consider an oriented link L composed of non intersecting component knots $\{K_i\}_{i=1}^r$. Assign an irrep ρ_i of the gauge group G to each K_i, then the *Wilson link* $W(L)$ is just the product over the Wilson loops associated to the knots

$$W(L) := \prod_{i=1}^{r} W_{\rho_i}(K_i) = \prod_{j=1}^{r} \operatorname{tr}_{\rho_j} \mathcal{P} \exp i \oint_{K_j} A. \tag{47}$$

Again, we are interested in the vev of $W(L)$,

$$\langle W(L) \rangle = Z^{-1} \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}A] \, \exp\big(ikS_{CS}[A]\big) \prod_{i=1}^{r} W_{\rho_i}(K_i). \tag{48}$$

For a Wilson link, reversing the orientation of a component K_i is akin to changing the associated representation ρ_i to its complex conjugate representation $\bar{\rho}_i$. In other words, to leave the vev the same, we have to simultaneously reverse the orientation and take the conjugate representation.

In the following sections, we will compute Equation (48) for $G = U(1)$ and $G = SU(2)$ in the fundamental representation, and see that they correspond to the writhe and Jones polynomial, respectively.

3.1 Abelian Chern–Simons

We study the case when $G = U(1)$ following [7]. In this case, the second term of the Chern–Simons doesn't appear since $A \wedge A$ vanishes, and we have

$$kS_{CS}[A] = \frac{k}{4\pi} \int_M A \wedge \mathrm{d}A \tag{49}$$

Recall our formula for linking number in terms of differential forms in Equation (6). We have

$$\mathrm{lk}(\gamma_1,\gamma_2) = \int_{x\in\Sigma_1} \delta_{\gamma_2}(x) = \int_M \delta_{\Sigma_1}(x)\wedge\delta_{\gamma_2}(x) = \int_M \delta_{\Sigma_1}(x)\wedge \mathrm{d}\delta_{\Sigma_2}(x) \tag{50}$$

which looks remarkable like our Chern–Simons action! This is not accidental. Consider the Wilson link in abelian Chern–Simons (47). To each irrep ρ_i of $U(1)$ we can associate an integer n_i. Furthermore, we lose the trace and the path ordering $\mathcal{P}$

$$W(L) = \prod_{i=1}^r W_{\rho_i}(K_i) = \prod_{j=1}^r \exp i\, n_j \oint_{K_j} A = \prod_{j=1}^r \exp i\, n_j \int_M \delta_{K_j}\wedge A = \exp i \int_M \sum_{i=1}^r (n_i\, \delta_{K_i})\wedge A. \tag{51}$$

where in the second to last step we wedged the δ-type 2-form δ_{K_i} with A so the integral is taken over all M. The space of connections $\mathcal{A}$ in this $U(1)$-bundle over M can be thought as the space of real-valued one forms on M, $\Omega_1(M)$. If θ is a function then under a gauge transformation $A\mapsto A+\mathrm{d}\theta$. Thus, the space of gauge transformations $\mathcal{G}$ is the space of exact 1-forms $\mathrm{d}\Omega_0(M)\subset\Omega_1(M)$. The vev is then

$$\langle W(L)\rangle = \frac{\int_{\Omega_0/\,\mathrm{d}\Omega_1}[\mathcal{D}A]e^{\frac{ik}{4\pi}\int_M A\wedge \mathrm{d}A + i\int_M \sum_{i=1}^r (n_i\,\delta_{K_i})\wedge A}}{\int_{\Omega_0/\,\mathrm{d}\Omega_1}[\mathcal{D}A]e^{\frac{ik}{4\pi}\int_M A\wedge \mathrm{d}A}} \tag{52}$$

which is quadratic in A. We know how to evaluate quadratic partition functions! Recall the n-dimensional Gaussian integral formula

$$\int \mathrm{d}^n x \exp\left(\frac{1}{2}\vec{x}\cdot A\vec{x} + \vec{J}\cdot\vec{x}\right) = \sqrt{\frac{(2\pi)^n}{\det(-A)}}\exp\left(-\frac{1}{2}\vec{J}\cdot A^{-1}\vec{J}\right). \tag{53}$$

We want to massage (52) to look like the Gaussian integral and apply the formula. In our case, we have $A=\mathrm{d}$ and $J=\sum_{i=1}^r \frac{2\pi}{k} n_i\,\delta_{K_i}$. Recall that $\mathrm{d}\delta_\Sigma = \delta_\gamma$ for some Seifert surface Σ of the loop γ, so $\mathrm{d}^{-1} J = \sum_{i=1}^r \frac{2\pi}{k} n_i\, \mathrm{d}^{-1}\delta_{K_i}$ formally exists. Applying the Gaussian integral formula, we get

$$\begin{aligned}\langle W(L)\rangle &= \exp\left(\frac{ik}{4\pi}\int_M -J\wedge \mathrm{d}^{-1} J\right)\\ &= \exp\left(\frac{i}{2}\sum_{i,j}^r n_i\,n_j \int_M -\delta_{K_i}\wedge \mathrm{d}^{-1}\delta_{K_j}\right)\\ &= \exp\left(\frac{i}{2}\sum_{i,j}^r n_i\,n_j \int_M \delta_{\Sigma_i}\wedge \mathrm{d}\mathrm{d}^{-1}(\mathrm{d}\delta_{\Sigma_j})\right)\\ &= \exp\left(\frac{i}{2}\sum_{i,j}^r n_i\,n_j \int_M \delta_{\Sigma_i}\wedge \mathrm{d}\delta_{\Sigma_j}\right)\end{aligned} \tag{54}$$

where we used $\mathrm{d}\delta_{\Sigma_i}=\delta_{K_i}$ and integrated by parts, then canceled the $\mathrm{d}\mathrm{d}^{-1}$ (note we can't cancel $\mathrm{d}^{-1}\mathrm{d}$, as d loses information). Now, split the sums into $i\neq j$ and $i=j$ parts. For the $i\neq j$ part, K_i, K_j are non intersecting. So we have

$$\langle W(L)\rangle = \exp\frac{i}{2}\left(\sum_{i\neq j}^r n_i\,n_j\,\mathrm{lk}(K_i,K_j) + \sum_i^r n_i^2\int_M \delta_{\Sigma_i}\wedge \mathrm{d}\delta_{\Sigma_i}\right). \tag{55}$$

The calculation of the second term is more subtle. It diverges in general since the Gauss linking integral (7) diverges as $\vec{x} \to \vec{y}$. The way to regularize it is by choosing a framing, which is called *topological regularization* in field theory. The result depends on the chosen framing, which is problematic since $\langle W(L) \rangle$ should be a physical quantity independent of framing. This is not a problem in our case since we know that in S^3 there exists a canonical choice of framing, the blackboard framing. It also turns out OK for Chern–Simons theory on a general 3-manifold because we know how the Wilson loop transforms under a change of framing[10]. In light of the formula for the writhe (14), we expect the second term to be the self-linking number for K_i.

To compute it, assume we have a framing for a knot K. We take the "ribbon" viewpoint of choosing a framing more seriously, and take K^f to be the knot displaced along the framing vector field (the other border of the ribbon). With a framing, the self-linking of K can be shown to be the linking number between K and K^f (see [2]). Hence, the Wilson link vev is

$$\langle W(L) \rangle = \exp \frac{i}{2} \left(\sum_{i \neq j}^{r} n_i \, n_j \, \mathrm{lk}(K_i, K_j) + \sum_{i}^{r} n_i^2 w(K) \right) = \exp \frac{i}{2} w(L) \Big|_{\rho_i \text{ fundamental}} . \tag{56}$$

So the vev of a Wilson link with fundamental representation in the $U(1)$ Chern–Simons theory turns out to be precisely the exponential of a classical knot invariant, the writhe!

3.2 Non-Abelian Chern–Simons

Now consider the case when $G = SU(n)$. The second term in the Chern–Simons action doesn't vanish, and thus we have a much more complicated interacting theory. To relate the $SU(2)$ CS theory to the Jones polynomial, we will try to reproduce its skein relation (20). Take the representations $\rho_i = \rho$ to be the fundamental representation. The Wilson link (48), reproduced here for convenience, is

$$\langle W(L) \rangle = Z^{-1} \int_{\mathcal{A}/\mathcal{G}} [\mathcal{D}A] \, \exp\left(ikS_{CS}[A]\right) \prod_{j=1}^{r} \mathrm{tr}_\rho \, \mathcal{P} \exp i \oint_{K_j} A. \tag{57}$$

Now focus on a crossing of L. Draw a sphere around the crossing as in Figure 17 (a), then we divide the manifold $M = S^3$ into two parts using the aforementioned *Heegaard splitting*. More precisely, we represent M as a connected sum of two manifolds M_R and M_L sharing a common boundary $\Sigma = S^2 - \{p_1, p_2, p_3, p_4\}$. The points $\{p_i\}$ are the points of intersection of the Wilson lines.

For the following, we use notation from [8]. Let $f : \Sigma \to \Sigma$ be an (orientation-reversing) homeomorphism, we write $M = M_R \cup_f M_L$, that is, we glue M_R and M_L using f as the identifying map. To each boundary we associate a Hilbert space $\mathcal{H}_R$ and $\mathcal{H}_L$. Since they have opposite orientation, we have $\mathcal{H}_R^* = \mathcal{H}_L$, that is, they are canonically dual Hilbert spaces. The map f can be represented by an operator $U_f : \mathcal{H}_R \to \mathcal{H}_L$. Moreover, recall from the discussion on quantization that $\dim \mathcal{H}_R = \dim \mathcal{H}_L = 2$. The "partial" path integral evaluated on M_R and M_L determines a state $|\psi\rangle = \langle W(L) \rangle \big|_{M=M_R} \in \mathcal{H}_R$ and a dual state $\langle \chi | = \langle W(L) \rangle \big|_{M=M_L} \in \mathcal{H}_L$ respectively. Using this and U_f, we can write the complete Wilson link vev (57) as

$$\langle W(L) \rangle = \frac{\langle \chi | U_f | \psi \rangle}{\langle 0 | U_f | 0 \rangle} \tag{58}$$

where $|0\rangle$ is the state with no Wilson loops.

We don't know how to evaluate this, since we don't know either vector. However, since $\mathcal{H}_R$ is a 2-dimensional vector space, if we have two other state vectors $|\psi'\rangle$, $|\psi''\rangle$, we have a linear relation

$$\alpha \, |\psi\rangle + \beta \, |\psi'\rangle + \gamma \, |\psi''\rangle = 0 \tag{59}$$

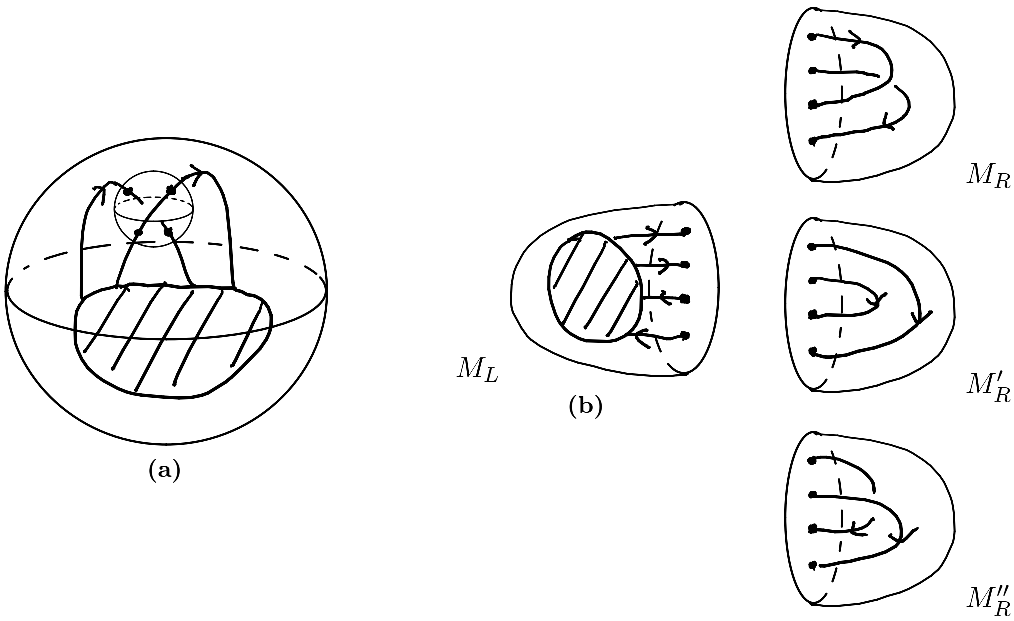


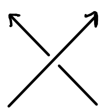
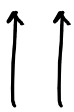
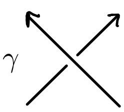
Figure 17: ***(a)*** *A link in $M = \mathbb{R}^3$ which is divided into two regions M_L and M_R. M_R is the interior of the small sphere which contains a crossing. M_L is the rest of M which contains a complicated link (shaded region).* ***(b)*** *The cut presented more explicitly, listed with the "replacement manifolds". Notice that M_R, M'_R and M''_R correspond to positive, zero, and negative crossing, respectively.*

where $\alpha, \beta, \gamma \in \mathbb{C}$. To obtain these other states, we can simply replace M_R with a suitable replacement M'_R or M''_R shown in Figure 17 (b) with appropriate orientation. Notice the similarity of this operation to skein relations! Now multiply by $\frac{\langle\chi|U_f}{\langle 0|U_f|0\rangle}$, we get

$$\frac{1}{\langle 0|U_f|0\rangle}\left(\alpha\,\langle\chi|U_f|\psi\rangle + \beta\,\langle\chi|U_f|\psi'\rangle + \gamma\,\langle\chi|U_f|\psi''\rangle\right) = 0 \tag{60}$$

$$\alpha\,\langle W(L)\rangle + \beta\,\langle W(L')\rangle + \gamma\,\langle W(L'')\rangle = 0 \tag{61}$$

where L' and L'' are links obtained from L by replacing M_R with M'_R and M''_R, respectively. Let's focus on the specific crossing and project this to a plane. We have

α  $+\,\beta$  $+\,\gamma$  $= 0$ (62)

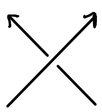
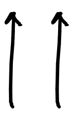
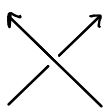
Witten explicitly evaluated α, β and γ in terms of n and k. The results are, after multiplying each by a convenient factor of $-\exp\left(\frac{\pi i(n^2-2)}{n(n+k)}\right)$ (which doesn't change the linear relation),

$$\alpha = \exp\left(\frac{n\pi i}{n+k}\right) \quad ; \quad \beta = \exp\left(\frac{-\pi i}{n+k}\right) - \exp\left(\frac{\pi i}{n+k}\right) \quad ; \quad \gamma = -\exp\left(\frac{-n\pi i}{n+k}\right). \tag{63}$$

Define a new variable

$$q := \exp\left(\frac{2\pi i}{n+k}\right),$$

we can then rewrite the skein relation (62) in terms of q. We get

$q^{\frac{n}{2}}$  $+\,(q^{-\frac{1}{2}} - q^{\frac{1}{2}})$  $-\,q^{-\frac{n}{2}}$  $= 0.$ (64)

Setting $n = 2$, we reproduce the skein relation for Jones polynomial (20)! If we don't set n to 2, we recover the skein relation for the HOMFLY polynomial which we briefly mentioned at the end of section 1.3. It turns out that using other groups and other representations will yield other invariant polynomials. In particular, $SO(n)$ and $Sp(2n)$ are related to the Kauffman polynomial (different from the Kauffman bracket). These all have defining skein relations, as discussed in [4].

4 Discussions

In our rush to arrive at link invariants from Chern–Simons theory, we have swept many details under the rug. We will attempt to partially rectify this unsatisfactory situation by providing some background on these details.

For example, it turns out that the Hilbert space $\mathcal{H}(\Sigma)$ in section 2.3 was previously described by conformal field theory. The states in $\mathcal{H}(\Sigma)$ are identified with the so called *conformal blocks* of the WZW model on Σ, which is an aspect of the AdS/CFT correspondence[9]. To understand this, we would have had carried out a *holomorphic quantization* by choosing a non-canonical complex structure on the phase space $\mathcal{M}$, the moduli space of flat connections. One difficult part of Witten's quantization was to show that the Hilbert space is in fact independent of this choice of complex structure.

Moreover, although we briefly discussed framing in the abelian case in section 3.1, we skimmed over many important features of framing dependence. In particular, the computation of α, β and γ in section 3.2, which we skipped, required a choice of framing. This is not so bad since in our restricted case we have a canonical choice of framing in S^3. Witten's formalism is powerful enough, however, to consider Wilson links over any general 3-manifold, where there might not be a canonical frame. Witten employs a technique known as *surgery* to reduce any 3-manifold to S^3. To understand Wilson links on a general 3-manifold, it suffices to understand how they transform under surgery. In addition to links requiring a framing, it turns out that the partition function $Z = \int [\mathcal{D}A]\, e^{ikS_{CS}}$ also depends on a *framing of the manifold* M, that is, a choice of equivalence class of trivializations of the bundle $TM \oplus TM$, and different choices shifts it by some factor. Atiyah showed that (referred to in [8]) any general M has a canonical choice of framing.

In section 2.3, we used a result known as *Heegaard splitting*. In 3.2, we talked about using Heegaard splitting to cut the manifold into two pieces without worrying too much about how the links or the partition function will behave, and just assumed that they behave nicely. In fact, since we worked on S^3, we only used a very limited version of the Heegaard splitting. The full result says that one can cut any 3-manifolds into two component pieces with a common boundary, where each piece is a *handlebody* of genus g (essentially g solid donuts $S^1 \times D^2$ attached together side by side). In our limited case, the Heegaard splitting can be represented as a connected sum. We can define a connected sum of manifolds, and similarly a connected sum of links. It's possible to show (again, see Witten[10] for details) that the partition function, the Wilson links, and even the invariant polynomials behaves in the nice multiplicative way under this operation. In fact, the axioms in the axiomatized formulation of TQFTs is actually based on the nice splitting of the manifold and the partition function[1],[5].

A Appendix

A.1 Proof of equivalence of linking numbers

We follow the proof in [3]. Consider a link with two components γ_1, γ_2. The Gauss linking number integral

$$\mathrm{lk}(\gamma_1, \gamma_2) = \oint_{\vec{x}\in\gamma_1} \oint_{\vec{y}\in\gamma_2} \frac{1}{4\pi} \frac{(\vec{x}-\vec{y}) \cdot \mathrm{d}\vec{x} \times \mathrm{d}\vec{y}}{|\vec{x}-\vec{y}|^3} \tag{65}$$

looks very hard to do. The key observation is that the integral is invariant under continuous formations of either curve as long as they don't cross, since the integral changes continuously but the linking number is integer-valued[5]. Zoom in on a crossing and deform the crossing such that the two strands are straight in some small cylinder, and are perpendicular to each other, as in Figure 18. Parametrize the curves with parameters s_x, s_y so that the strands enter at cylinder at $s_x = s_y = -1$ and leave at $s_x = s_y = +1$. Moreover, we impose that $|\dot{\vec{x}}(s_x)| = |\dot{\vec{y}}(s_y)| = 1$.

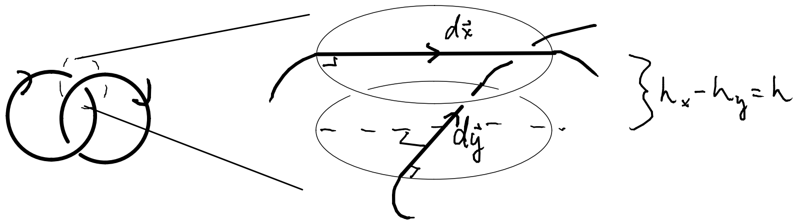


Figure 18: *Computing the Gauss integral over a crossing.*

Then, within the small cylindrical region, we have

$$\begin{aligned} \vec{x} &= (s_x, 0, h_x) & \dot{\vec{x}}(s_x) &= (1,0,0) & \mathrm{d}\vec{x} &= \dot{\vec{x}}\,\mathrm{d}s_x \\ \vec{y} &= (0, s_y, h_y) & \dot{\vec{y}}(s_y) &= (0,1,0) & \mathrm{d}\vec{y} &= \dot{\vec{y}}\,\mathrm{d}s_y \end{aligned} \tag{66}$$

and the contribution of this crossing, say c, to the Gauss integral (7) becomes

$$I(c) = \int_{-1}^{1} \mathrm{d}s_x \int_{-1}^{1} \mathrm{d}s_y \frac{1}{4\pi} \frac{h}{(h^2 + s_x^2 + s_y^2)^{3/2}} = \frac{1}{\pi} \arctan \frac{1}{h\sqrt{2+h^2}} \tag{67}$$

where $h = h_x - h_y$. In a link diagram, we take the projection onto a plane. This corresponds to taking the limit $h \to 0$. If the $\mathrm{d}\vec{x}$ strand lies over the $\mathrm{d}\vec{y}$ strand, we have $I(c) \to \frac{1}{2}$, which corresponds to a positive crossing. On the other hand, if the $\mathrm{d}\vec{x}$ strand lies under the $\mathrm{d}\vec{y}$ strand, we have $I(c) \to -\frac{1}{2}$, corresponding to a negative crossing. Furthermore, outside the crossing regions, the integral vanishes in the limit since the triple product is taken over three coplanar vectors. Hence, if we integrate over the entire link, we have

$$\mathrm{lk}(\gamma_1, \gamma_2) = \oint_{\vec{x}\in\gamma_1} \oint_{\vec{y}\in\gamma_2} \frac{1}{4\pi} \frac{(\vec{x}-\vec{y}) \cdot \mathrm{d}\vec{x} \times \mathrm{d}\vec{y}}{|\vec{x}-\vec{y}|^3} = \sum_{c\in\text{crossings}} I(c) = \sum_{c\in\text{crossings}} \frac{1}{2}\,\mathrm{sign}(c) \tag{68}$$

as required.

[5] We don't know this a priori, but the linking number can be viewed as the degree of some map, which is integer-valued.

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