Black Hole Thermodynamics and Hawking Radiation

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Abstract

Black holes are interesting objects to study in physics because they marry many notions of quantum mechanics and general relativity. Studying black holes provides insight into a yet unknown theory of quantum gravity. We find that the laws of mechanics of black holes are closely related to the laws of thermodynamics. The discovery of Hawking radiation will allow the formulation of these relationships into physical correspondences. We will discuss the thermodynamics of black holes and how the Unruh effect and Hawking radiation arise from considering quantum field theory.

I. Introduction

The existence of black hole solutions to Einstein's field equations (EEF) has been known since the theory of general relativity was formulated. It was only over the past thirty years that physicists realized a deep connection between the laws that describe black holes and the classical laws of thermodynamics. Bekenstein pointed out that the Hawking area theorem, which states that the area of a black hole does not decrease, is analogous to the second law of thermodynamics, which states that the entropy of a closed system will not decrease[3]. This was formalized to a general second law which we will discuss in section III. Later, Bardeen, Carter, and Hawking showed the four laws of black hole mechanics[2] and pointed out that these are very similar to the four laws of thermodynamics. We can identify $E \leftrightarrow M$, $T \leftrightarrow C\kappa$, and $S \leftrightarrow \frac{A}{8\pi C}$ where M is the mass, κ the surface gravity, A the area of a black hole and C some unidentified constant. This was viewed as a mathematical curiosity at first. However, the discovery of Hawking radiation showed that black holes radiate at a temperature of $T = \frac{\kappa}{2\pi}$, which shows that the correspondence is physical.

We will provide a mostly qualitative description of the thermodynamics of black holes to provide some physical intuition. Readers interested in a more detailed discussion can consult our references. We will follow both the review article by Ross[10] and the book by Carroll[4] in this paper. Our convention for the signature of the Minkowski metric is $(+---)$. We will work in natural units $G = k_B = \hbar = c = 1$.

II. Black Holes

In the following section, we will briefly review some metric solutions for black holes. We will not derive the solutions from GR because it doesn't offer too much in the way of understanding the physics.

The simplest of the black hole solutions is the Schwarzschild black hole. It is a *stationary, asymptotic flat, static* and *vacuum* solution to EFE. The Schwarzschild black hole is the most ordinary black hole; it has mass but neither charge nor angular momentum. It is described by the Schwarzschild metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 - r^2 \, d\Omega^2. \tag{1}$$

There is a coordinate singularity at $r = 2M$. To eliminate the coordinate singularity, recall the Kruskal coordinates

$$U = \left(\frac{r}{2M} - 1\right)^{-1} e^{(r+t)/4M} \tag{2}$$

$$V = \left(\frac{r}{2M} - 1\right)^{-1} e^{(r-t)/4M}. \tag{3}$$

and the metric in terms of U, V,

$$ds^2 = -\frac{32M^3}{r} e^{-r/2M}\, dU\, dV + r^2\, d\Omega \tag{4}$$

Note that we can further "compress" the Kruskal coordinates from infinity down to a finite size by setting

$$u' = \arctan U \tag{5}$$

$$v' = \arctan V \tag{6}$$

which will give us the Penrose diagram.

The concept of horizons will be important for our discussion. First, there is of course the event horizon, where classically nothing can escape after crossing. There is also the concept of a *Killing horizon*. Consider the Schwarzschild metric. $K^\mu = \partial_t^\mu$ is a Killing vector for the metric since no component depends explicitly on t. Outside the event horizon, K^μ is timelike. Inside the event horizon, K^μ becomes spacelike. At the event horizon K^μ will be null. In general, we call null hypersufaces[1] where Killing vectors are null *Killing horizons*. Note that the Killing vector K^μ will be normal to the Killing horizon. The notion of Killing horizons is independent from that of event horizons. However, given some reasonable conditions, it is possible to classify event horizons as follows:

1. In a stationary, asymptotically flat spacetime, every event horizon is a Killing horizon for some Killing vector field K^μ.
2. If the spacetime is static, then $K^\mu = \partial_t^\mu$, which represents time translations at infinity.
3. If the spacetime is stationary but not static, then $K^\mu = \partial_t^\mu + \Omega \partial_\phi^\mu$, where ∂_ϕ is a rotational Killing vector field and Ω is some constant.

For a Killing horizon Σ, we can define a quantity κ called the *surface gravity* by

$$K^\mu \nabla_\mu K^\nu = \kappa K^\nu \tag{7}$$

or equivalently

$$\kappa^2 = \frac{1}{2}(\nabla^\mu K^\nu)(\nabla_\mu K_\mu) \tag{8}$$

As an example, consider the Kruskal coordinates for the Schwarzschild metric[5]. $K = \partial_t = -\frac{1}{4M} U \partial_U + \frac{1}{4M} V \partial_V$ is a Killing vector for the metric. Define two hypersurfaces $N^+ = \{(U, V, \theta, \phi) : U = 0\}$ and $N^- = \{(U, V, \theta, \phi) : V = 0\}$. These are both Killing horizons of K since K is normal to them. The surface gravity is $\frac{1}{4M}$ on N^+ and $-\frac{1}{4M}$ on N^-.

[1]That is, a hypersurface whose tangent spaces are space-like except for one one which corresponds to the world line of a particle moving at the speed for light. For example, a light cone is a null hypersurface

This suggests that we should interpret the surface gravity as the acceleration of a static observer near the horizon as measured by a static observer at infinity. This turns out to be the correct interpretation in a static asymptotically flat spacetime. The surface gravity is also intimately related to the temperature of a black hole, as we will see in section III.

To make the definition of a Killing horizon unique, we have to normalize Killing vectors. For a static, asymptotically flat spacetime like the Schwarzschild metric, we can set

$$K_\mu K^\mu (r \to \infty) = 1. \tag{9}$$

We sometimes call the norm of the Killing field $V = \sqrt{K_\mu K^\mu}$ ranging between 0 and 1 the *redshift factor*. It relates the emitted and observed frequencies of a photon measured by a static observer. It can be shown that $\kappa = Va$ at the horizon where a is the acceleration that an observer needs to have to remain static.

We can also define something called a *bifurcate Killing horizon*: it is a pair of Killing horizons with the same Killing vector which intersect over a spacelike two-surface called the *bifurcation surface*. In the case of the Schwarzschild metric, $N = N^+ \cup N^-$ is a bifurcate Killing horizon with a bifurcation two-sphere $N^+ \cap N^- = \{(U, V, \theta, \phi) : U = V = 0\}$.

When the spacetime is stationary, but not static, then $K^\mu = \partial_t^\mu$ is still a Killing vector. However, K^μ will not be null at a Killing horizon but at some timelike surface outside the horizon. We call the surface the *ergosurface*. Inside this surface, observers cannot remain stationary. These observers have to move with respect to the Killing field, but not necessarily towards to event horizon.

This will give us a way to extract energy from a stationary but not static black hole in a process called the *Penrose process*[8]. Consider the Kerr metric defining a rotating but uncharged black hole.

$$\begin{aligned} ds^2 = \left(1 - \frac{2Mr}{\rho^2}\right) dt^2 + \frac{2Mar\sin^2\theta}{\rho^2}(dt\, d\phi + d\phi\, dt) \\ + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right) d\phi^2 \end{aligned} \tag{10}$$

where $\Delta = r^2 - 2Mr + a^2$, $\rho^2 = r^2 + a^2\cos^2\theta$, and $a = \frac{J}{M}$ is the angular momentum per unit mass. In general, this metric will allow an outer and an inner event horizon just like the Reissner-Nordström metric.

We have two Killing vectors $K^\mu = \partial_t^\mu$ and $R^\mu = \partial_\phi^\mu$. Let $p^\mu = m\frac{dx^\mu}{d\tau}$ be the four momentum of some particle. Then we can obtain the conserved energy and angular momentum of the particle as

$$E = K_\mu p^\mu = m\left(1 - \frac{2Mr}{\rho^2}\right)\frac{dt}{d\tau} + \frac{2mMar}{\rho^2}\sin^2\theta\frac{d\phi}{d\tau} \tag{11}$$

$$L = -R_\mu p^\mu = -\frac{2mMar}{\rho^2}\sin^2\theta\frac{dt}{d\tau} + \frac{m(r^2+a^2)^2 - m\Delta a^2\sin^2\theta}{\rho^2}\sin^2\theta\frac{d\phi}{d\tau} \tag{12}$$

The sign difference comes from the fact that K^μ is timelike while R^μ is spacelike. Inside the ergosphere, K^μ becomes spacelike and hence we could have

$$E = K_\mu p^\mu < 0. \tag{13}$$

Now consider a particle with momentum p^μ which travels inside the ergosphere and decays into two particles with momentum p_1^μ and p_2^μ. By conservation of momentum, we have $p^\mu = p_1^\mu + p_2^\mu$ and hence

$$E = E_1 + E_2. \tag{14}$$

If the particle decays in a way such that (13) is realized for particle 2, and particle 1 escapes the ergosphere, then we will have

$$E_1 > E \tag{15}$$

And hence particle 1 can leave with more energy then it entered with. This energy comes from the

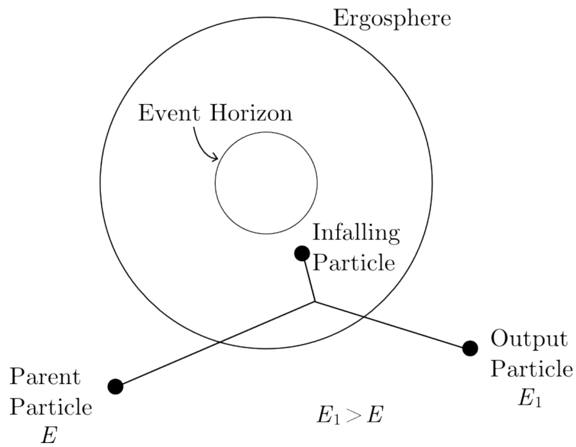

Figure 1: *Penrose process.*

rotational energy of the black hole. To see this, consider the linear combination $\chi^\mu = K^\mu + \Omega R^\mu$ which generates the outer event horizon. It is possible to interpret Ω as the angular velocity of the horizon. If particle 2 crosses the outer event horizon by moving forward in time, we have

$$p_{2_\mu} \chi^\mu \geq 0 \tag{16}$$

and hence

$$p_{2_\mu} \chi^\mu = p_{2_\mu} K^\mu + \Omega p_{2_\mu} R^\mu = E_2 - \Omega L_2 \geq 0 \implies \tag{17}$$

$$0 > \frac{E_2}{\Omega} \geq L_2 \tag{18}$$

since E_2 was negative. Hence, particle 2 has opposite angular momentum of the black hole. Let us consider the efficiency of this process. Consider the change in mass and angular momentum of the black hole. We must have

$$\delta M = E_2 \tag{19}$$

$$\delta J = L_2 \tag{20}$$

where $J = Ma$ is the angular momentum of the black hole. Hence, by (18), we must have

$$\Omega\, \delta J \leq \delta M \tag{21}$$

we obtain equality in the limit when p_2^μ is tangent to χ^μ. In this case, the area of the black hole will not increase since $A = 4\pi(r_+^2 + a^2)$ where $r_+ = M + \sqrt{M^2 - a^2}$ is the radius of the outer horizon[9]. We can show that in general, we have

$$\delta M = \frac{\kappa}{8\pi}\, \delta A + \Omega\, \delta J \tag{22}$$

where κ is the surface gravity of the Killing horizon.

III. Classical Black Hole Thermodynamics

Equation (22) looks a lot like the first law of thermodynamics! In fact, it is possible to show that in general, for a rotating charged black hole, we have

$$dM = \frac{\kappa}{8\pi}\, dA + \Omega\, dJ + \Phi\, dQ. \tag{23}$$

where Ω is the angular velocity of the outer event horizon and Φ is the electric potential. Equation (23) is indeed the first law of black hole mechanics. Let us verify the first law for the Schwarzschild solution as an example. We have $dJ = dQ = 0$. The area of the Schwarzschild black hole is $A = 4\pi r_s^2 = 4\pi(2M)^2 = 16\pi M^2$. Hence, if we change M by a small amount dM, we change the area by $16\pi(M + dM)^2 \simeq 16\pi M^2 + 32\pi M\, dM = A + 32\pi M\, dM$. Recall that $\kappa = \frac{1}{4M}$, hence we have $dA = \frac{8\pi}{\kappa}\, dM$.

Is it possible to generalize the first law to an arbitrary Lagrangian theory of gravity[17],[7]. Consider a gravitational Lagrangian

$$\mathbf{L} = \mathbf{L}(g_{\alpha\beta}; R_{\alpha\beta\gamma\delta}, \nabla_\lambda R_{\alpha\beta\gamma\delta}, \ldots; \phi, \nabla_\lambda \phi, \ldots) \tag{24}$$

where ϕ is the collection of all matter fields of the theory and ... represents the finitely many covariant derivatives of the Riemann tensor and of ϕ. We can construct an $(n-1)$-form Noether current $\mathbf{J}$ associated to a vector field X. This current will give an $(n-2)$-form Noether charge $\mathbf{Q}$ which satisfies $\mathbf{J} = d\mathbf{Q}$ when the equations of motion are satisfied. From this, we can consider a stationary black hole solution of the theory and an associated bifurcate Killing horizon $\mathcal{H}$. It turns out that if we define the entropy as the integral over the bifurcation surface Σ of $\mathcal{H}$ as

$$S = 2\pi \int_\Sigma \mathbf{Q}[X'] = 2\pi \int_\Sigma \mathbf{X}^{cd} \boldsymbol{\epsilon}_{cd} \tag{25}$$

where X' is the Killing field which vanishes on Σ, $\mathbf{X}^{cd}$ an $(n-2)$-form, and $\boldsymbol{\epsilon}_{cd}$ a volume element on the tangent space perpendicular to Σ, then we get the more general first law of thermodynamics

$$\delta\mathcal{E} = \frac{\kappa}{2\pi}\, \delta S + \, \delta W \,(\text{work terms}) \tag{26}$$

where $\mathcal{E}$ is an energy defined by an integral at infinity and κ is the usual surface gravity.

We can obtain the $(n-2)$-form $\mathbf{X}^{cd}$ by taking

$$\mathbf{X_{ab}} = -\frac{\delta \mathbf{L}}{\delta \mathbf{R_{ab}}} \tag{27}$$

where $\mathbf{R_{ab}}$ is the Ricci tensor. In the case of the Einstein-Hilbert action, $\mathbf{X_{ab}}$ will be proportional to the area element of the horizon and the entropy S will work out to be $\frac{A}{4}$.

We can apply this formalism to various theories. For example, it was shown that this formula for the entropy agrees with the microscopic calculation of M-theory[16].

Now we consider Hawking's area theorem for a black hole. The area theorem states: *Assuming the cosmic censorship principle and the weak energy condition ($T_{\mu\nu}t^\mu t^\nu \geq 0$ for energy momentum tensor $T_{\mu\nu}$ and timelike vectors t^μ), the area of a spatial event horizon in an asymptotically flat spacetime is non-decreasing.*

This theorem, combined with equation (23), suggests that we identify $T\, dS$ in the first law of thermodynamics with $\frac{\kappa}{8\pi}\, dA$ in the first law of black hole mechanics. In section V.2, we will show that black holes radiate at a temperature $T = \frac{\kappa}{2\pi}$. Hence, we can identify $S \leftrightarrow \frac{A}{4}$. While it seems like Hawking's area theorem can be taken as the second law of black hole mechanics, it turns out that Hawking radiation will allow the black hole to evaporate and lose mass, and hence lose area. This is because by including quantum fluctuations, we will violate the weak energy condition which was used to prove the area theorem. However, Bekenstein proposed a generalized second law which combines the entropy of matter and black holes. It states that, under a variety of assumptions, the combined entropy of matter outside the black hole and the entropy of the black hole will always be non-decreasing. That is,

$$\delta\left(S+\frac{A}{4}\right)\geq 0. \tag{28}$$

In statistical mechanics, we identify entropy with the logarithm of the number of quantum states. However, the no hair theorem suggests that black holes of fixed mass, spin, and charge only have one state. This inconsistency could be solved by string theory by considering the degenerate states in string theory[12].

We can also prove a zeroth law of thermodynamics, which states that the surface gravity κ is constant over the event horizon of a *stationary* black hole. We showed this for the Schwarzschild black hole. A non-stationary black hole such as one which recently formed from gravitational collision doesn't necessarily have constant κ. This is of course reminiscent of the zeroth law of thermodynamics, which states that systems in equilibrium share the same temperature.

IV. Quantum Field Theory

We will develop some tools and terminology need to to discuss to Unruh effect and the Hawking radiation in this section. Since both of these are relativistic quantum effects, we must take quantum field theory into account[2]. Recall the flat spacetime Klein-Gordon equation

$$\ddot{\phi}-\nabla^2\phi+m^2\phi=\Box\phi+m^2\phi=0 \tag{29}$$

where $\Box$ is the d'Alembertian operator. The Klein-Gordon equation is the field theoretic version of the equation of motion of a harmonic oscillator. We can write down a plane wave solution

$$\phi(x^\mu)=\phi_0 e^{ik_\mu x^\mu}=\phi_0 e^{iwt-i\vec{k}\cdot\vec{x}} \tag{30}$$

where $\omega^2=\vec{k}^2+m^2$. An important distinction from the harmonic oscillator is that now we get a set of solutions parametrized by the wavevector $\vec{k}$ and the sign of ω. It is possible to write down the most general solution by constructing a set of complete orthonomal solutions with respect to the inner product

$$(\phi_1,\phi_2)=-i\int_{\Sigma_t}\left(\phi_1\partial_t\phi_2^*-\phi_2^*\partial_t\phi_1\right)d^{n-1}x \tag{31}$$

where Σ_t is a constant time hypersurface. The inner product will be independent of Σ_t by Stokes' theorem and Klein-Gordon equation. This gives us a orthonormal set of modes

$$f_{\vec{k}}(x^\mu)=\frac{e^{ik_\mu x^\mu}}{\left((2\pi)^{n-1}2\omega\right)^{1/2}}. \tag{32}$$

[2] We should use a theory of quantum gravity in principle. However, it is unknown how to formulate such a theory.

We call these $f_{\vec{k}}$ modes *positive frequency.* That is, $\partial_t f_{\vec{k}} = i\omega f_{\vec{k}}$ for $\omega > 0$. We can get a set of solution modes by taking the complex conjugate $f^*_{\vec{k}}$. The complex conjugate modes are said to be *negative frequency* since $\partial_t f_{\vec{k}} = -i\omega f_{\vec{k}}$ for $\omega > 0$. The modes $f_{\vec{k}}$ and $f^*_{\vec{k}}$ form a complete set.

We can now proceed to canonical quantization. Let $\pi = \dot{\phi}$ be the conjugate momentum. Instead of classical variables, we can view ϕ and π as operators $\hat{\phi}$ and $\hat{\pi}$. Taking the hint from a harmonic oscillator, we postulate the following canonical commutation relations on a hypersurface Σ_t:

$$\begin{aligned} [\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{x}')] &= 0 \\ [\hat{\pi}(t,\vec{x}), \hat{\pi}(t,\vec{x}')] &= 0 \\ [\hat{\phi}(t,\vec{x}), \hat{\pi}(t,\vec{x}')] &= i\delta^{(n-1)}(\vec{x}-\vec{x}'). \end{aligned} \tag{33}$$

Just like the quantum harmonic oscillator, we can define annihilation and creation operators $\hat{a}^\dagger_{\vec{k}}$ and $\hat{a}_{\vec{k}}$ as the decomposition

$$\hat{\phi} = \int d^{n-1}k \, \left(\hat{a}_{\vec{k}} f_{\vec{k}}(t,\vec{x}) + \hat{a}^\dagger_{\vec{k}} f^*_{\vec{k}}(t,\vec{x}) \right) \tag{34}$$

These operators obey the familiar commutation relations as one can check by plug in into equation (33). Just as there are an infinite number of $f_{\vec{k}}$, there are an infinite number of annihilation and creation operators indexed by $\vec{k}$. There is one single vacuum state $|0\rangle$ defined as

$$\hat{a}_{\vec{k}} |0\rangle = 0 \tag{35}$$

for all $\vec{k}$. We also have states $|n_{\vec{k}}\rangle$ of $n_{\vec{k}}$ with the same momenta $\vec{k}$. These are produced by repeatedly applying the creation operator to the vacuum state. If we have different momenta $\vec{k}_i$, then we have

$$|n_1, n_2, \ldots, n_k\rangle = \frac{1}{\sqrt{n_1! n_2! \ldots n_k!}} \left(\hat{a}^\dagger_{\vec{k}_1} \right)^{n_1} \left(\hat{a}^\dagger_{\vec{k}_2} \right)^{n_2} \ldots \left(\hat{a}^\dagger_{\vec{k}_k} \right)^{n_k} |0\rangle . \tag{36}$$

We can define a number operator for each $\vec{k}$, $\hat{n}_{\vec{k}_i} := \hat{a}^\dagger_{\vec{k}_i} \hat{a}_{\vec{k}_i}$ which satisfies $\hat{n}_{\vec{k}_i} |n_1, n_2, \ldots, n_k\rangle = n_i |n_1, n_2, \ldots, n_k\rangle$. These eigenstates of the number operators form a basis called the *Fock basis* for the complete Hilbert space. We call a Hilbert space with a Fock basis a *Fock space.*

We could do the same analysis in curved spacetime by suitably changing partial derivative to covariant derivative and inserting $\sqrt{-g}$ and g_{ab} where needed. However, since there is no timelike Killing vector in general, we will not be able to separate solutions into time dependent factors and space dependent factors. Hence, in general, we cannot classify modes as positive or negative frequency. We can still find a set of basis modes, but there will be many such sets and we have no reason to prefer one set over another. Hence, the vacuum and the number operator will depend on the set we will choose.

Let $f_i(x^\mu)$ be such a set with corresponding annihilation and creation operators $\hat{a}_i$, $\hat{a}^\dagger_i$ and number operator $\hat{n}_{f_i} = \hat{a}^\dagger_i \hat{a}_i$ and let $g_i(x^\mu)$ be another set with $\hat{b}_i$, $\hat{b}^\dagger_i$ and $\hat{n}_{g_i} = \hat{b}^\dagger_i \hat{b}_i$. We can express one set of modes as a linear combination of the other (and its complex conjugate) as follows

$$g_i = \sum_j \left(\alpha_{ij} f_j + \beta_{ij} f^*_j \right) \tag{37}$$

$$f_i = \sum_j \left(\alpha^*_{ji} f_j - \beta_{ji} g^*_j \right) . \tag{38}$$

The transformation from one set of modes to another is called the *Bogolubov transformation* and α_{ij}, β_{ij} are the *Bogolubov coefficients*. We can use the Bogolubov coefficients to transform the annihilation and creation operators like the modes. Consider the expectation value of $\hat{n}_{g_i}$ in the vacuum of f_i.

$$\begin{aligned}
\langle 0_f | \hat{n}_{g_i} | 0_f \rangle &= \langle 0_f | \hat{b}_i^\dagger \hat{b}_i | 0_f \rangle \\
&= \langle 0_f | \sum_{jk} \left(\alpha_i j \hat{a}_j^\dagger - \beta i j \hat{a}_j \right) \left(\alpha_i^* k \hat{a}_k - \beta^* i k \hat{a}_k^\dagger \right) | 0_f \rangle \\
&= \sum_{jk} \beta_{ij} \beta_{ik}^* \langle 0_f | \hat{a}_j \hat{a}_k^\dagger | 0_f \rangle \\
&= \sum_{jk} \beta_{ij} \beta_{ik}^* \langle 0_f | \left(\hat{a}_k^\dagger \hat{a}_j + \delta_{jk} \right) | 0_f \rangle \\
&= \sum_j \beta_{ij} \beta_{ij}^* = \sum_j |\beta_{ij}|^2
\end{aligned} \tag{39}$$

Hence, if any of the β_{ij} is non zero, the vacuum states $|0_f\rangle$ and $|0_g\rangle$ will differ. we will see the physical manifestation of this phenomenon in the next section.

V. Quantum Black Hole Thermodynamics

In section III, taking Hawking radiation into account allowed us to conclude that there is a physical interplay between black hole mechanics and thermodynamics since we could identify $T \leftrightarrow \frac{\kappa}{2\pi}$ and $S \leftrightarrow \frac{A}{4}$. Without the Hawking radiation tempeture, we wouldn't have know the exact relationship between $T\,dS$ and $\frac{\kappa}{8\pi}\,dA$. That is, without Hawking radiation, the analogy between black hole mechanics and thermodynamics remains an analogy.

In this section, we will use the tools we developed in section IV to derive the Unruh effect. We will then use the Unruh effect to argue for the existence of Hawking radiation.

V.1 Unruh Effect

Consider a uniformly accelerating observer in a two-dimensional Minkowski spacetime with acceleration α. The coordinates x^μ is given by $(\frac{1}{\alpha}\sinh(\alpha\tau), \frac{1}{\alpha}\cosh(\alpha\tau))$. Hence, the acceleration vector a^μ is $\frac{D^2 x^\mu}{d\tau^2} = \frac{d^2 x^\mu}{d\tau^2} = (\alpha\sinh(\alpha\tau), \alpha\cosh(\alpha\tau))$ since the covariant derivative is equal to the usual derivative in flat spacetime. We will introduce a set of coordinates called *Rindler coordinates*[11]. It is analogous to polar coordinates in Minkowski spacetime. Let

$$t = \frac{1}{a} e^{a\xi} \sinh(a\eta) \tag{40}$$

$$x = \frac{1}{a} e^{a\xi} \cosh(a\eta) \tag{41}$$

and $x > |t|$. In these coordinates, the metric is

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2), \tag{42}$$

and the acceleration vector is $a^\mu = (a^\eta, a^\xi) = (\frac{\alpha}{a}\tau, \frac{1}{a}\ln\frac{a}{\alpha})$. So for a uniformly accelerating observer with $\alpha = a$ we have $a^\mu = (\tau, 0)$. We call such an observer a *Rindler observer*. The causal structure of this space is very similar to that of the Schwarzschild solution for the region $r > 2M$. Moreover, we

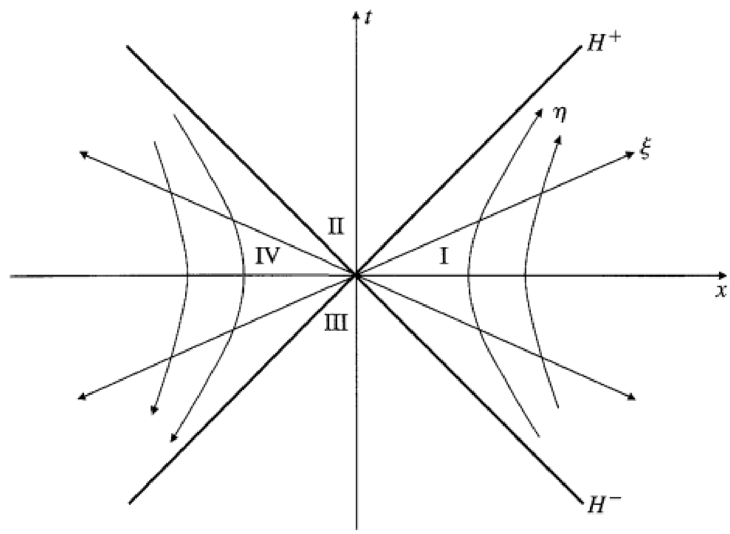

Figure 2: *Rindler space. $H^{\pm}$ are the Killing horizons. (Carroll p.404)*

can define regions I, II, III, and IV like the Schwarzschild solution. Observers in region I are causally disconnected from observers in region IV. Our coordinates only covers region I for a positively accelerated Rindler observer. We can define

$$t = -\frac{1}{a}e^{a\xi}\sinh(a\eta) \tag{43}$$

$$x = -\frac{1}{a}e^{a\xi}\cosh(a\eta) \tag{44}$$

and $x < |t|$ to cover region IV.

Since the metric components doesn't depend on η, we automatically get a Killing vector $\partial_\eta = a(x\partial_t + t\partial_x)$. The surfaces $x = 0$ and $t = 0$ are Killing horizons of ∂_η with surface gravity $\kappa = a$. Along the surface $t = 0$, ∂_η is timelike. Hence, we can use it to define positive and negative frequency modes and construct a Fock basis. We need to consctruct this for region I and IV separately but they will have the same vacuum state $|0_R\rangle$.

Now suppose that $|0_M\rangle$ is the Minkowski vacuum state. Then the expectation value $\langle 0_M|\, \hat{n}_R^{(1)}\, |0_M\rangle$ of the number operation in the Rindler space for region I will turn out to be nonzero. It is not a trivial task to calculate the Bogolukov coefficients β_{ij}. We will simply cite the result here[11],[4].

$$\langle 0_M|\, \hat{n}_R^{(1)}\, |0_M\rangle = \frac{1}{e^{2\pi\omega/a} - 1}\delta(0) \tag{45}$$

where the delta function $\delta(0)$ is an artifact of the choice of the basis modes. This tell us that a Rindler observer will see a blackbody spectrum of particles with temperature

$$T = \frac{a}{2\pi} = \frac{\kappa}{2\pi}. \tag{46}$$

This thermal emission is the *Unruh effect.* To find the temperature of another uniformly accelerated observer (at constant ξ), we can related the emitted frequencies using the redshift factor as

$$\omega_2 = \frac{V_1}{V_2}\omega_1 = e^{a(\xi_1 - \xi_2)}\omega_1. \tag{47}$$

Recall $V = e^{a\xi}$ is the magnitude of the Killing vector. Hence, if $\xi_1 = 0$, the observer at $\xi_2 = \xi$ will see a redshifted temperature of $T = \frac{ae^{-a\xi}}{2\pi}$.

There seems to be a paradox. We have $\langle T_{\mu\nu} \rangle = 0$ and $\langle T_{\mu\nu} \rangle$ is a physical quanity and shouldn't depend on the coordinate system. Therefore, the Rindler observer shouldn't observe particles since there is no energy-momentum. This is not a contracdiction, however. Consider the Rindler observer who is carrying a particle detector with him. To keep the detector accelerating uniformly, the observer must perform constant work. In the frame of the Minkowski observer, we will see that the particle detector both emits and absorbs particles. The take away lesson from the Unruh effect is that vacuum and particles are *observer dependent* concepts and not fundamental concepts.

V.2 Hawking Radiation

Based on the Unruh effect, it is easy to see how the Hawking effect arises for a Schwarzschild black hole. Consider a stationary observer at a fixed radius r outside the black hole. To remain stationary, the acceleration of the observer is

$$a = \frac{M}{r^2\sqrt{1 - \frac{2M}{r}}}. \tag{48}$$

For r near the Schwarzschild radius, a is very large and hence the associated time scale a^{-1} is very small compared to $2M$. We can neglect the curvature of spacetime on this small timescale.

Now consider a freely-falling observer and assume that the scalar field ϕ is regular near the horizon and is in a vacuum state as seen by this observer. By the equivalence principle, this observer will see the stationary observer accelerating uniformly at a and hence detect Unruh radiation at $T = \frac{a}{2\pi}$.

To calculate the temperature of another observer who's not near the horizon (hence whose time scale might be too large to ignore the curvature of spacetime), we can relate the temperature by redshift factors like for the Unruh effect as

$$T_2 = \frac{V_1}{V_2} T_1 = \frac{V_1}{V_2}\frac{a}{2\pi}. \tag{49}$$

Taking $V_2 \to 1$ at infinity and $V_1 a \to \kappa$ as $r \to 2M$, we get that $T = \frac{\kappa}{2\pi}$ where $\kappa = \frac{1}{4M}$ is the surface gravity. Hence, observers at infinity will see thermal radiation from a black hole emitted at $T = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}$. This is the *Hawking effect.*

We made a crucial assumption that the scalar field ϕ is regular near the horizon in our analysis. There exists such a state called the *Hartle-Hawking vacuum.* There also exists vacuum state for which this is not true called the *Boulware vacuum.* In this state, the accelerated observer sees no particle at all.

Our analysis was done for an eternal Schwarzschild black hole. Hawking's original description of the effect was for black holes formed through gravitational collapse. We could consider a spacetime that is nearly flat in the null past infinity and Schwarzschild in the null future infinity. We will find the same answer for temperature as above after some algebra. We could also derive Hawking radiation for more general black hole solutions such as the Kerr[1] and Kerr-Newman solutions[15].

While the black radiates, energy conservation implies that it'll lose mass at a rate given by the Stefan-Boltzmann law

$$P \sim AT^4 \sim M^2 M^{-4} \sim M^{-2} \sim -\frac{dM}{dt}. \tag{50}$$

Integrating gives the black hole a life time on the order of M^3. If the sun was a black hole of the same mass, we get a lifetime of about 10^{74} s. Compare this to the age of the universe, which is only about 10^{17} s!

Hawking radiation leads to a unsolved paradox called the *information loss paradox*. Consider two different initial states. If we collapse them into black holes of the same mass, charge, and spin, they will both radiate the same spectrum of particles. Hence, the information about the states before the collapse has been irretrievably lost, that is, unitarity of the system is violated. There is a conjectured solution to the paradox called the *black hole complementarity* hypothesis, formulated by Susskind, Thorlacius, Uglum[13], and 't Hooft[14]. It suggests that information is both reflected by and passes through the horizon, and no observer can confirm both so that the no-cloning theorem of quantum mechanics is not violated. This hypothesis is supported by a recent paper due to Hayden and Preskill[6]. If we assume that black holes dynamics are unitary and reaches equilibrium quickly (that is, reaches temperature $T = \frac{\kappa}{2\pi}$ in some upper-bounded time), then black holes will act as some kind of information mirror, reflecting information thrown inside in some amount of time.

VI. Conclusion

We have shown that there are close analogies between the classical laws of black holes and the laws of thermodynamics. By studying quantum field theory in curved spacetime, we were able to derive the Unruh effect and the Hawking effect. The Unruh effect shows that the concept of vacuum and particles are observer dependent. The Hawking effect shows that black holes radiate at a temperature proportional to the surface gravity of the event horizon. By applying the Hawking radiation temperature, we turned the analogies to thermodynamics into physical correspondences. There are some puzzles which remain such as the information loss paradox. The study of these puzzles may provide insight into a theory of quantum gravitation.

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