Clifford Algebra

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Introduction

In 1878, Clifford introduced his "geometric" algebra in order to generalize the notion of complex numbers and quaternions. Today, Clifford algebra has found applications in multiple fields such as geometry, physics, and image processing. Notably, in quantum field theory, Dirac's gamma matrices which are used to solve the Dirac equation, form a Clifford algebra.

In this paper, we introduce Clifford algebras of finite dimensional vector fields. Both the real and complex Clifford algebras have surprising periodicity in their dimensions, which is a specific case of *Bott periodicity*. We will classify real and complex Clifford algebras using this periodicity. We also explore the representation theory of complex Clifford algebras, which turns out to be simple because the classification of complex Clifford algebras is simple. We will touch upon the construction of the so called spin groups, $\mathrm{Spin}(n)$, which are the non-trivial double cover of $SO(n)$. We will follow the seminar notes of Park [3], the course notes of Borcherds [1], and the course notes of Woit [5].

1 Algebras

Before moving on to Clifford algebra, we need to first define what an algebra is. We will only consider fields of characteristic zero throughout the paper. It's fairly straightforward to extend the analysis any field which doesn't have characteristic 2. Fields of characteristic 2 are more tricky but still possible. Moreover, unless otherwise noted, we assume our algebras and vector spaces are finite dimensional. Again, it's possible to extend the analysis to infinite dimensional cases, but it will not be covered in this paper.

Definition 1. Let K be a field. A ring R is called a *K-algebra* if there exists a map $\cdot : K \times R \to R$ such that

1. $(R, +)$ is a vector space over K.
2. $k \cdot (rs) = (k \cdot r)s = r(k \cdot s)$ for all k in K and r, s in R.

An algebra that is a finite dimensional vector space over K is called a *finite dimensional algebra*, and similarly for an infinite dimensional vector space.

In some sense, a K-algebra is a K-vector space with a "vector multiplication". We can extend K-vector spaces to K-algebras via the tensor product.

Definition 2. Let V and W be vector spaces over the field K. The *tensor product* of V and W is the vector space

$$V \otimes W = \mathrm{Span}\{v \otimes w : v \in V, w \in W\}.$$

with properties

1. $(v + v') \otimes w = v \otimes w + v' \otimes w$
2. $v \otimes w + w' = v \otimes w + v \otimes w'$
3. $k(v \otimes w) = (kv) \otimes w = v \otimes (kw)$

for $v, v' \in V$, $w, w' \in W$, and $k \in K$.

When it's clear, we will write $v \otimes w$ as vw.

Note that if $\{v_i\}_{i \in I}$ is a basis for V and $\{w_j\}_{j \in J}$ is a basis for W, then $\{v_i \otimes w_J\}_{i \in I, j \in J}$ is a basis for $V \otimes W$.

Property 3. looks similar to the property required for an algebra, but it doesn't quite work. However, we can construct a very large algebra using the tensor product,

Definition 3. Let V be a vector space over the field K. Let $\bigotimes^i V$ denote the tensor product of V with itself i times. Set $\otimes^0 V = K$. The tensor algebra is the K-algebra

$$T(V) = \bigoplus_{i=0}^{\infty} \left(\bigotimes^i V \right),$$

where we identify $k \otimes v_1 \otimes v_2 \otimes \ldots \otimes v_n$ with $kv_1 \otimes v_2 \otimes \ldots \otimes v_n$ for any finite set of vectors $v_1, \ldots, v_n$.

The tensor algebra $T(V)$ is the *free* algebra on V. It is an infinite dimensional algebra. We will construct a finite dimensional algebra, the Clifford algebra, out of it. For this, we need the extra structure of a *symmetric bilinear form* on the vector field.

Definition 4. Let V be a vector space over the field K. A *symmetric bilinear form* is a function $\langle \cdot, \cdot \rangle : V \times V \to K$ such that

1. $\langle v, w \rangle = \langle w, v \rangle$
2. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
3. $\langle kv, w \rangle = k \langle v, w \rangle$

In general, symmetric bilinear forms are not inner products. For example, when $K = \mathbb{C}$, the inner product is conjugate symmetric and not symmetric. With this, we can finally define a Clifford algebra.

2 Clifford algebra

Definition 5. Let V be a vector space over a field K with a symmetric bilinear form $\langle \cdot, \cdot \rangle$. Let $T(V)$ be the tensor algebra over V and let $I(V)$ be the ideal in $T(V)$ generated by $\{v \otimes v + \langle v, v \rangle 1_K : v \in V\}$ where 1_K is the identity in K viewed as a subalgebra of $T(V)$. The *Clifford algebra* of V is the quotient algebra

$$\mathrm{C}\ell(V) \equiv T(V)/I(V).$$

Note that the Clifford algebra depends on the choice of the symmetric bilinear form, so it's not unique for a given V.

To be more comfortable with this definition, we will go through a few examples.

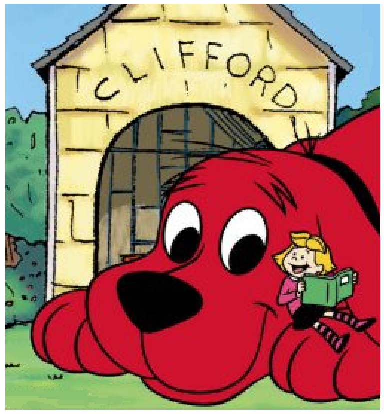

Figure 1: *A picture of Clifford, the big red algebra.*

Example 6. Let $\mathbb{R}$ be a one-dimensional vector space over itself and let $a \in \mathbb{R}$ be non zero, then $\{a\}$ forms a basis for $\mathbb{R}$. The tensor algebra $T(\mathbb{R})$ looks like

$$T(\mathbb{R}) = \bigoplus_{i=0}^{\infty} \left(\bigotimes^{i} \mathbb{R} \right) = (\mathbb{R}) \oplus (\mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R}) \oplus (\mathbb{R} \otimes \mathbb{R} \otimes \mathbb{R}) \oplus \dots$$

Define a symmetric bilinear form on $\mathbb{R}$ as $\langle a, a \rangle = 1$. Modding out by $\{a \otimes a + \langle a, a \rangle 1 : a \in \mathbb{R}\}$ "collapses"[1] all the even and odd tensor powers of $\mathbb{R}$ in $T(\mathbb{R})$ respectively, so $\mathrm{C}\ell(\mathbb{R}) = T(\mathbb{R})/I(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}$ as vector spaces. Furthermore, imposing the relation $a \otimes a + \langle a, a \rangle 1 = 0$ gives us $a \otimes a = -1$ or simply $a^2 = -1$ in $\mathrm{C}\ell(\mathbb{R})$.

Combining these, we see that elements of $\mathrm{C}\ell(\mathbb{R})$ can be written as $x + ya$ for $x, y \in \mathbb{R}$ and

$$\begin{aligned}(x + ya) + (u + va) &= (x + u) + (y + v)a \\ (x + ya)(u + va) &= (xu - yv) + (xv + yu)a.\end{aligned}$$

From these, we can see that $\mathrm{C}\ell(\mathbb{R}) \cong \mathbb{C}$ as algebras over $\mathbb{R}$.

Example 7. With the same set up as in the previous example, but this time let's define our symmetric bilinear form on $\mathbb{R}$ as $\langle a, a \rangle = -1$. Then imposing the relation $a \otimes a + \langle a, a \rangle 1 = 0$ gives us $a^2 = 1$ in $\mathrm{C}\ell(\mathbb{R})$. We get the addition and multiplication rules in $\mathrm{C}\ell(\mathbb{R})$ as

$$\begin{aligned}(x + ya) + (u + va) &= (x + u) + (y + v)a \\ (x + ya)(u + va) &= (xu + yv) + (xv + yu)a.\end{aligned}$$

Here, $\mathrm{C}\ell(\mathbb{R}) \cong \mathbb{R}^2$ as algebras with the isomorphism $\phi(x + ya) = \big((x + y), (x - y)\big)$ and multiplication in $\mathbb{R}^2$ taken as componentwise multiplication.

Example 8. There is nothing that restricts us from setting the symmetric bilinear form to 0 so consider the form on $\mathbb{R}$, $\langle a, a \rangle = 0$. Then imposing the relation $a \otimes a + \langle a, a \rangle 1 = 0$ gives us $a^2 = 0$ in $\mathrm{C}\ell(\mathbb{R})$. The addition and multiplication rules in $\mathrm{C}\ell(\mathbb{R})$ are then

$$\begin{aligned}(x + ya) + (u + va) &= (x + u) + (y + v)a \\ (x + ya)(u + va) &= (xu) + (xv + yu)a.\end{aligned}$$

[1] In a very loose sense. But we think it helps to visualize what modding out by that ideal actually means.

It turns out that $\mathrm{C}\ell(\mathbb{R}) \cong \bigwedge \mathbb{R}$, the *exterior algebra* of $\mathbb{R}$. This is true in general: if we set $\langle\cdot,\cdot\rangle = 0$ in V, then $\mathrm{C}\ell(V) \cong \bigwedge V$. The study of exterior algebras is an interesting subject of its own but we'll not venture further into it.

Let us investigate the basis and the dimension of the Clifford algebra of a vector space V. Suppose V is finite dimensional with basis $\{v_1, v_2, \ldots, v_n\}$. Consider the definition of the Clifford algebra 5. We only need to specify the generating set of the ideal $I(V) = \big\langle v \otimes v + \langle v, v\rangle 1_K : v \in V \big\rangle$ over the basis elements of V to specify the ideal, that is, $I(V) = \big\langle v_i \otimes v_i + \langle v_i, v_i\rangle 1_K : 1 \leq i \leq n \big\rangle$. The basis of the Clifford algebra will be the tensor products of the basis elements of V where no one basis element occur twice (otherwise it gets "killed" by the relation we impose). So the Clifford algebra has basis

$$
\begin{aligned}
&\{e_{i_1} e_{i_2} \ldots e_{i_k} : 1 \leq i_1 < i_2 < \ldots < i_k \leq n \text{ and } 0 \leq k \leq n\} && (1)\\
=&\{1_K\} \cup \{e_i : 1 \leq i \leq n\} \cup \{e_i e_j : 1 \leq i < j \leq n\} \cup \ldots \cup \{e_1 e_2 \ldots e_n\} && (2)
\end{aligned}
$$

For each value of k, there are n choose k basis elements. The dimension of the Clifford algebra is hence

$$\dim \mathrm{C}\ell(V) = \sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Since the definition of $I(V)$ only depended on the bilinear form $\langle v, v\rangle$ of a single vector v and not of two different vectors, we could have defined it using a *quadratic form*. A quadratic form is a polynomial where each term has degree 2. For example, $q\big((x, y)\big) = 3x^2 + 2xy - 8y^2$ is a quadratic form on $\mathbb{R}^2$. In fact, to each symmetric biliner form we can associate a quadratic form $q : V \to K$, $q(v) = \langle v, v\rangle$ for $v \in V$. The definition of $\mathrm{C}\ell(V)$ will then only depend on the choice of q. Moreover, given a quadratic form q, we can get the symmetric bilinear form back via the *polariztion identity*

$$\langle v, w\rangle = \frac{1}{2}\left(q(v + w) - q(v) - q(w)\right).$$

Some readers will recognize that this is the same as the parallelogram law for norms and inner products.

2.1 Real Clifford algebras

Let's focus on the case where $K = \mathbb{R}$. It turns out that quadratic form associated to a symmetric bilinear form can always be diagonalized. This is formalized in the following theorem.

Theorem 9. *Let V be an n-dimensional vector space over $\mathbb{R}$ with a bilinear form q associated to a symmetric bilinear form $\langle\cdot,\cdot\rangle$, then q is diagonalizable. In particular, there exists non-negative integers p, q and a basis $\{e_i\}_{1\leq i\leq n}$ of V such that*

$$q(v) = q(x_1 e_1 + x_2 e_2 + \ldots + x_n e_n) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2$$

If $p + q = n$, we say that the quadratic form q is non-degenerate. The pair (p, q) is then called the *signature* of q. We will mostly be interested in non-degenerate quadratic forms in this paper.

This theorem is a consequence of the spectral theorem in linear algebra since we can associate the symmetric bilinear form $\langle v, v\rangle$ to $v^T A v$ where A is a symmetric matrix and symmetric matrices are diagonalizable. This will help us classify Clifford algebras of $V = \mathbb{R}^n$.

2.2 Classification of real Clifford algebras

Definition 10. Let p, q be non negative integers such that $p+q=n$. Let $C\ell_{p,q}(\mathbb{R})$ be the Clifford algebra of $\mathbb{R}^n$ equipped with the quadratic form

$$q(v) = q(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_n^2$$

We want to find the structure of these $C\ell_{p,q}(\mathbb{R})$. Let $\{e_i\}$ be the standard basis for $\mathbb{R}^n$ and let $C\ell_{p,q}(\mathbb{R})$ be its Clifford algebra. Then

$$\begin{aligned} e_ie_j + e_je_i &= e_ie_j + e_je_i + e_i^2 + e_j^2 - e_i^2 - e_j^2 \\ &= -q(e_i+e_j) + q(e_i) + q(e_j) \\ &= -2\langle e_i, e_j\rangle \\ &= \begin{cases} -2\delta_{ij} & \text{if } i \le p \\ 2\delta_{ij} & \text{if } i > p \end{cases} \end{aligned}$$

This gives us another way to characterize the relations that we impose on the tensor algebra. In particular, we have to specify how basis elements anticommute. From this, we can see that 1 was indeed a basis for $C\ell_{p,q}(\mathbb{R})$ and that it has dimension 2^n. Now Example 6 tells us that $C\ell_{1,0}(\mathbb{R}) \cong \mathbb{C}$ and Example 7 tells us that $C\ell_{0,1}(\mathbb{R}) \cong \mathbb{R} \oplus \mathbb{R}$. Let's check for higher p and q.

1. $C\ell_{2,0}(\mathbb{R})$ has the following relations on the basis vectors (obtained by modding out the ideal): $e_1^2 + 1 = 0, e_2^2 + 1 = 0, e_1e_2 + e_2e_1 = 0$. There is an isomorphism between $C\ell_{2,0}(\mathbb{R})$ and the quaternions $\mathbb{H}$ given by $e_1 \mapsto i$, $e_2 \mapsto j$. This induces that $e_1e_2 \mapsto k$. Let us check: $(e_1e_2)^2 = e_1e_2e_1e_2 = e_1(-e_1e_2)e_2) = -(e_1e_1)(e_2e_2) = -(-1)(-1) = -1$ as required. The other properties follow similarly.

2. $C\ell_{1,1}(\mathbb{R})$ has the following relations on the basis vectors: $e_1^2+1=0, e_2^2-1=0, e_1e_2+e_2e_1=0$. There is an isomorphism between $C\ell_{1,1}(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ given by $e_1 \mapsto \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $e_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

3. Similarly, $C\ell_{0,2}(\mathbb{R})$ has the following relations on the basis vectors: $e_1^2 - 1 = 0, e_2^2 - 1 = 0, e_1e_2+e_2e_1 = 0$. There is an isomorphism between $C\ell_{0,2}(\mathbb{R})$ and $M_{2\times 2}(\mathbb{R})$ given by $e_1 \mapsto \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $e_2 \mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.

This seem to become very tedious and hard to compute for any larger p, q. Thankfully, that is not the case. Let $K = \mathbb{R}$ and let's try to compute $C\ell(V \oplus W)$.

Let $\{e_i\}_{1\le i\le n}$ and $\{f_i\}_{1\le i\le m}$ be orthogonal basis for V and W, respectively. Assume that $\dim(V) = n$ is even. Set $g_i = e_1e_2\ldots e_nf_i$. The relation between e_i and g_j is

$$e_ig_j = e_ie_1e_2\ldots e_nf_j = e_1e_2\ldots e_nf_je_i = (-1)^n g_je_i = g_je_i$$

since n was even and e_i anticommutes with everything except itself. Now,

$$\begin{aligned} g_ig_j &= g_ie_1e_2\ldots e_nf_j \\ &= e_1e_2\ldots e_ng_if_j \\ &= e_1e_2\ldots e_ne_1e_2\ldots e_nf_if_j \\ &= e_1e_2\ldots e_ne_1e_2\ldots e_n(-f_jf_i) \\ &= -e_1e_2\ldots e_ne_1e_2\ldots e_nf_jf_i \\ &= -g_jg_i \end{aligned}$$

And

$$\begin{aligned} g_i g_i &= e_1 e_2 \dots e_n e_1 e_2 \dots e_n f_i f_i \\ &= (-1)^{n(n-1)/2} e_1^2 e_2^2 \dots e_n^2 f_i^2 \\ &= (-1) e_1^2 e_2^2 \dots e_n^2 f_i^2 \text{ since } n \text{ is even} \\ &= \pm 1 \end{aligned}$$

So the g_i's satisfy the relation of a Clifford algebra. They also commute with e_i's. Hence, $\mathrm{C}\ell(V \oplus W) \cong \mathrm{C}\ell(V) \otimes \mathrm{C}\ell(W')$, where W' is W with its quadratic form $q_W \to (-1)e_1^2 e_2^2 \dots e_n^2 \cdot q_W = q_{W'}$. Take $\dim V = n = 2$. This gives us a theorem that relates different $\mathrm{C}\ell_{p,q}(\mathbb{R})$'s.

Theorem 11. *For all non-negative integers p, q, we have*

$$\mathrm{C}\ell_{p+2,q}(\mathbb{R}) \cong \mathbb{H} \otimes \mathrm{C}\ell_{q,p}(\mathbb{R}) \tag{3}$$

$$\mathrm{C}\ell_{p+1,q+1}(\mathbb{R}) \cong M_{2\times 2}(\mathbb{R}) \otimes \mathrm{C}\ell_{p,q}(\mathbb{R}) \tag{4}$$

$$\mathrm{C}\ell_{p,q+2}(\mathbb{R}) \cong M_{2\times 2}(\mathbb{R}) \otimes \mathrm{C}\ell_{q,p}(\mathbb{R}) \tag{5}$$

where the indices switch if $e_1^2 \dots e_n^2 = 1$.

We can now classify some of the $\mathrm{C}\ell_{p,q}(\mathbb{R})$'s.

$p\backslash q$	0	1	2
0	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$M_{2\times 2}(\mathbb{R})$
1	$\mathbb{C}$	$M_{2\times 2}(\mathbb{R})$	
2	$\mathbb{H}$		

The diagonal is easy to fill since by (4), we just need to tensor with $M_{2\times 2}(\mathbb{R})$ to move down right diagonally. The off diagonal entries require a bit more effort to find.

This result is very nice, but it still seem to leave us with infinitely many real Clifford algebras to classify. This is fortunately not the case (again).

Corollary 12 (Bott periodicity)**.** *For all non-negative integers p, q, we have*

$$\begin{aligned} \mathrm{C}\ell_{p+8,q}(\mathbb{R}) &\cong M_{16\times 16}(\mathbb{R}) \otimes \mathrm{C}\ell_{p,q}(\mathbb{R}) \\ \mathrm{C}\ell_{p,q+8}(\mathbb{R}) &\cong M_{16\times 16}(\mathbb{R}) \otimes \mathrm{C}\ell_{p,q}(\mathbb{R}) \\ \mathrm{C}\ell_{p+4,q+4}(\mathbb{R}) &\cong M_{16\times 16}(\mathbb{R}) \otimes \mathrm{C}\ell_{p,q}(\mathbb{R}) \end{aligned}$$

where the tensor product is over $\mathbb{R}$, the base field.

Hence, we only have to fill a 8×8 table and then extend by Bott periodicity. We will not present the entire table but we invite the reader to take a look at [2]. The classification of real Clifford algebra follows.

Theorem 13 (Classification of real Clifford Algebras)**.** *The Clifford algebra $\mathrm{C}\ell_{p,q}(\mathbb{R})$ is isomorphic to the following associative algebra:*

$p - q \mod 8$	$\mathrm{C}\ell_{p,q}(\mathbb{R})$
$0, 6$	$M(2^{(p+q)/2}, \mathbb{R})$
7	$M(2^{(p+q-1)/2}, \mathbb{R}) \oplus M(2^{(p+q-1)/2}, \mathbb{R})$
$1, 5$	$M(2^{(p+q-1)/2}, \mathbb{C})$
$2, 4$	$M(2^{(p+q-2)/2}, \mathbb{H})$
3	$M(2^{(p+q-3)/2}, \mathbb{H}) \oplus M(2^{(p+q-3)/2}, \mathbb{H})$

2.3 Complex Clifford algebras

We can also look at complex Clifford algebras, which differ from the real Clifford algebras we've seen in a few aspects.

Definition 14. Let A be an algebra over $\mathbb{R}$. The *complexification* of A is the $\mathbb{C}$-algebra

$$A \otimes_{\mathbb{R}} \mathbb{C} = \{x + yi : x, y \in A, i \text{ the imaginary unit}\}$$

with operations:

1. $(x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i$
2. $(a + bi)(x + yi) = (ax - by) + (ay + bx)i$
3. $(x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i$

Note that if we dropped 3., then the definition holds for complexification of vector spaces.

Let V be a vector space over $\mathbb{R}$ with the symmetric bilinear product $\langle \cdot, \cdot \rangle$. We can extend $\langle \cdot, \cdot \rangle$ to the complexification of V by bilinearity:

$$\langle v_1 + w_1 i, v_2 + w_2 i\rangle = \langle v_1, v_2\rangle - \langle w_1, w_2\rangle + \langle v_1, w_2\rangle i + \langle w_1, v_2\rangle i. \tag{6}$$

From this, we can show that complexification commutes with "taking the Clifford algebra"[2].

Lemma 15. *Let V be a vector space over $\mathbb{R}$, then $\mathrm{C}\ell(V \otimes_{\mathbb{R}} \mathbb{C}) \cong \mathrm{C}\ell(V) \otimes_{\mathbb{R}} \mathbb{C}$.*

We postpone the proof a little bit. For now, let's look at what happens to the quadratic form q associative to a symmetric bilinear form $\langle \cdot, \cdot \rangle$ of V when we complexify V. Let $\{e_i\}$ be a basis for V and suppose that q is non-degenerate, then there exists non-negative integer p such that

$$q(x_1 e_1 + x_2 e_2 + \ldots + x_n e_n) = x_1^2 + x_2^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_n^2,$$

where $x_i \in \mathbb{R}$. The set $\{e_i\}$ also forms a basis for $V \otimes_{\mathbb{R}} \mathbb{C}$ by construction of the complexification. Consider the quadratic form $q_{\mathbb{C}} : V \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C}$ associated to the extended symmetric bilinear form defined in (6). We have

$$q_{\mathbb{C}}(z_1 e_1 + z_2 e_2 + \ldots + z_n e_n) = z_1^2 + z_2^2 + \ldots + z_p^2 - z_{p+1}^2 - \ldots - z_n^2,$$

where $z_i \in \mathbb{C}$. However, the set $\{e_1, \ldots, e_p, ie_{p+1} \ldots, ie_n\}$ is also a basis for $V \otimes_{\mathbb{R}} \mathbb{C}$. Plugging into the extended symmetry bilinear product, we get that

$$q_{\mathbb{C}}(z_1 e_1 + z_2 e_2 + \ldots + z_n e_n) = z_1^2 + z_2^2 + \ldots + z_p^2 + z_{p+1}^2 + \ldots + z_n^2,$$

that is, it makes all the signs positive and hence the signature (p, q) is irrelevant for a complexified real Clifford algebra. Therefore, we have the following isomorphisms:

$$\mathrm{C}\ell_{n,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{C}\ell_{n-1,1}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \ldots \cong \mathrm{C}\ell_{0,n}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Now let us return to the proof of the lemma.

Proof. We know that $V \cong R^n$ for some n. It's easy to see that $R^n \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}^n$. Let $p + q = n$, we want to show that $\mathrm{C}\ell_{p,q}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C} \cong \mathrm{C}\ell(\mathbb{C}^{p+q})$. Since the signature doesn't matter, We simply need to show that there is a set $\{f_1, f_2, \ldots, f_n\}$ where every pair anti-commutes and $f_i^2 = -1$. However, $\{e_1 \otimes_{\mathbb{R}} 1, \ldots, e_p \otimes_{\mathbb{R}} 1, e_{p+1} \otimes_{\mathbb{R}} i, \ldots, e_n \otimes_{\mathbb{R}} i\}$ is such a set, where $\{e_i\}$ is a basis for V. $\square$

We will denote the Clifford algebras isomorphic to $\mathrm{C}\ell_{n,0}(\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ by $\mathrm{C}\ell_n(\mathbb{C})$ since there is no ambiguity. Let's attempt to classify the complex Clifford algebras.

[2] More formally, the $\mathrm{C}\ell$ functor commutes with the complexification functor.

2.4 Classification of complex Clifford algebras

It turns out that complex Clifford algebras are easier to classify than real Clifford algebras. This is not surprising since complex Clifford algebras don't have a signature on their quadratic forms. Let's check for small n's.

1. For $\mathrm{C}\ell_0(\mathbb{C})$, we get $\mathrm{C}\ell_0(\mathbb{C}) \cong \mathbb{R} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C}$.

2. For $\mathrm{C}\ell_1(\mathbb{C})$, we get $\mathrm{C}\ell_1(\mathbb{C}) \cong \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$. This is actually isomorphic to $\mathbb{C}^2$ via the isomorphism $\phi : C^2 \to \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ with $\phi(1, 0) = \frac{1}{2}(1 \otimes 1 + i \otimes i)$ and $\phi(0, 1) = \frac{1}{2}(1 \otimes 1 - i \otimes i)$ which determines ϕ.

3. For $\mathrm{C}\ell_2(\mathbb{C})$, we get $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$ which can be shown to be isomorphic to $M_{2\times 2}(\mathbb{C})$.

However, it turns out we don't even need 3. because complex Clifford algebras has a periodicity of 2, as shown by the following theorem.

Theorem 16 (Bott periodicity for complex Clifford algebras)**.** *For all $n \geq 0$, there is an isomorphism of complex associative algebras*

$$\mathrm{C}\ell_{n+2}(\mathbb{C}) \cong \mathrm{C}\ell_n(\mathbb{C}) \otimes_{\mathbb{C}} M_{2\times 2}(\mathbb{C}).$$

Note the the tensor product is over $\mathbb{C}$.

Proof. Write $\mathbb{C}^{n+2} = \mathbb{C}^n \oplus \mathbb{C}e_1 \oplus \mathbb{C}e_2$ and define a complex linear map

$$\phi : \mathbb{C}^{n+2} \to \mathrm{C}\ell_n(\mathbb{C}) \otimes_{\mathbb{C}} M_{2\times 2}(\mathbb{C})$$

via

$$\begin{aligned} \phi(v) &= v \otimes \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \\ \phi(e_1) &= 1 \otimes \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \\ \phi(e_2) &= 1 \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \end{aligned}$$

for all $v \in \mathbb{C}^n$. Then similarly to what we did for $\mathrm{C}\ell(V \oplus W) \cong \mathrm{C}\ell(V) \otimes \mathrm{C}\ell(W')$ in the real case, we can show that these $\phi(v), \phi(e_1), \phi(e_2)$ satisfy the relation of a Clifford algebra. The induced map $\Phi : \mathrm{C}\ell_{n+2}(\mathbb{C}) \to \mathrm{C}\ell_n(\mathbb{C}) \otimes_{\mathbb{C}} M_{2\times 2}(\mathbb{C})$ is then injective. Since the dimensions are equal, Φ is an isomorphism. □

As a corollary, we get the classification of complex Clifford algebras.

Corollary 17 (Classification of complex Clifford algebras)**.** *For all $n \geq 0$, the complex Clifford algebra $\mathrm{C}\ell_n(\mathbb{C})$ is isomorphic to the following associative algebra*

parity of n	$\mathrm{C}\ell_n(\mathbb{C})$
n even	$M(2^{n/2}, \mathbb{C})$
n odd	$M(2^{(n-1)/2}, \mathbb{C}) \oplus M(2^{(n-1)/2}, \mathbb{C})$

3 Representations of Clifford algebras

We will discuss the representations of Clifford algebras. The representation theory of Clifford algebra is important for physics, and in particular, Quantum Field Theory. There is a very nice paper [4] which cover it in depth. We will go through some basic results here. We assume again that vector spaces be finite dimensional, by it is possible to extend to infinite dimensional cases.

Definition 18. Let V be a vector space over a field K. The set $\mathrm{Hom}(V, V)$ is the set of K-vector space homomorphisms (i.e. K-linear maps) from V to V.

$\mathrm{Hom}(V, V)$ form an algebra over K under pointwise addition and composition.

Definition 19. A *representation* of an algebra A on a vector space V over the field K is a homomorphism of algebras

$$\rho : A \to \mathrm{Hom}(V, V).$$

A representation of an algebra A on a vector space V over K makes V into an A-module via $a \cdot v \equiv \rho(a)(v)$ where $a \in A$ and $v \in V$. If A is a Clifford algebra, we call the module operation *Clifford multiplication.*

Definition 20. Let ρ, ρ' be representations of an algebra A on vector spaces V and V' over K, respectively. We say that ρ and ρ' are *equivalent* if there exists an isomorphism of vector spaces $\phi : V \to V'$ such that $\rho'(a)(v') = (\phi \circ \rho(a) \circ \phi^{-1})(v')$ for all $v' \in V'$ and a in A.

Equivalence of representation is, of course, an equivalence relation as can be easily checked.

Definition 21. Let A be an algebra on a vector space V over K with a representation ρ. A vector subspace W of V is called *invariant* if $\rho(a)(W) \subset W$ for every $a \in A$. A representation is called *reducible* if there are proper invariant subspaces, that is, $W \neq V, \{0\}$. A representation that is not reducible is called *irreducible.*

Definition 22. A representation of an algebra A on a vector space V over K is *completely reducible* if there exists subspaces $V_1, V_2, \ldots, V_r$ of V such that $V \cong \bigoplus_{i=1}^{r} V_i$ where V_i's are irreducible.

Theorem 23 (Complete reducibility)**.** *Every finite dimensional representation of a K-algebra can be expressed as a direct sum of irreducible representations.*

Hence, to study the representations of a K-algebra, we only need to focus on the irreducible representations. There are not many irreducible representations of complex Clifford algebras because of their simple classification. Recall that $\mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^n) \cong M(n, \mathbb{C})$.

Theorem 24. *Let $n \geq 1$. Up to equivalence, the only irreducible representation of $M(n, \mathbb{C})$ is $M(n, \mathbb{C})$ on $\mathbb{C}^n$. For $M(n, \mathbb{C}) \oplus M(n, \mathbb{C})$, there are two equivalence classes of irreducible representations: $\rho_i : M(n, \mathbb{C}) \oplus M(n, \mathbb{C}) \to M(n, \mathbb{C})$ with $\rho_1(A_1, A_2) = A_1$ and $\rho_2(A_1, A_2) = A_2$.*

We can write down these representation explicitly. Consider the case where $n = 2m$ is even. Then we have an representation $\rho_n : \mathrm{C}\ell_n(\mathbb{C}) \to M(2^m, \mathbb{C})$. First, when $m = 1$, define $\rho_2 : \mathrm{C}\ell_2(\mathbb{C}) \to M(2, \mathbb{C})$ as

$$\rho_2(e_1) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\rho_2(e_2) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

where e_i's are the standard basis elements of $\mathbb{C}^2$. We can now construct ρ_n inductively. Let $\{e_1, e_2, \ldots, e_n, e_{n+1}, e_{n+2}\}$ be the standard basis for $\mathbb{C}^{n+2}$ and suppose we know ρ_n. Set

$$\rho_{n+2}(e_i) = \begin{pmatrix} 0 & \rho_n(e_i) \\ \rho_n(e_i) & 0 \end{pmatrix}$$

for $1 \leq i \leq n$ and define

$$\rho_{n+2}(e_{n+1}) = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$$

$$\rho_{n+2}(e_{n+2}) = \begin{pmatrix} 0 & 0 & iI & 0 \\ 0 & 0 & 0 & -iI \\ iI & 0 & 0 & 0 \\ 0 & -iI & 0 & 0 \end{pmatrix}$$

each ρ_n is injective. Counting the dimensions gives us that ρ_n is actuall an isomorphism when n is even. Note that this is similar to the construction we did to prove Bott periodicity for complex Clifford algebras.

Now consider the case where $n = 2m+1$ is odd. In this case, a representation of $C\ell_n(\mathbb{C})$ is an homomorphism of algebra that maps into $M(2^m, \mathbb{C})$. For $m = 0$, define $\rho(e_1) = -i$. We can define ρ_n inductively. Let $\{e_1, e_2, \ldots, e_n, e_{n+1}\}$ be the standard basis for $\mathbb{C}^{n+1}$ and suppose we know ρ_n. Set

$$\rho_{n+1}(e_i) = \rho_n(e_i)$$

for $1 \leq i \leq n$ and define

$$\rho_{n+1}(e_{n+1}) = \begin{pmatrix} -iI & 0 \\ 0 & iI \end{pmatrix}.$$

4 Spin Groups

Clifford algebras can be used to construct the so called spin groups $\mathrm{Spin}(n)$ which are non-trivial double covers of $SO(n)$. There are multiple ways to do so. We will construct the Lie algebra of $\mathrm{Spin}(n)$ in terms of elements of the Clifford algebra.

The Lie algebra $\mathfrak{so}(n)$ consists of n by n antisymmetric matrices with real entries. There is a basis given by

$$L_{ij} = E_{ij} - E_{ji}$$

for $i < j$ where E_{ij} denotes the square matrix with 1 in it's i, j^{th} position and 0 elsewhere. L_{ij} generators rotations in the ij-plane. Moreover, they satisfy the Lie brackets:

$$[L_{ij}, L_{kl}] = \delta_{il} L_{kj} - \delta_{ik} L_{lj} + \delta_{jl} L_{ik} - \delta_{jk} L_{il}.$$

Now, the basis $\{e_i\}$ of the Clifford algebra $C\ell_{n,0}(\mathbb{R})$ satisfy

$$e_i e_j + e_j e_i = -2\delta ij.$$

It's a straightforward computation to show that $\frac{1}{2} e_i e_j$ will satisfy the same commutation relation as L_{ij}, that is,

$$\left[\frac{1}{2} e_i e_j, \frac{1}{2} e_k e_l\right] = \delta_{il}\left(\frac{1}{2} e_k e_j\right) - \delta_{ik}\left(\frac{1}{2} e_l e_j\right) + \delta_{jl}\left(\frac{1}{2} e_i e_k\right) - \delta_{jk}\left(\frac{1}{2} e_i e_l\right).$$

From these relationships, we can see that there is an isomorphism between the elements of the form $e_ie_j, i < j$ of $C\ell_{n,0}(\mathbb{R})$ and the Lie algebra $\mathfrak{so}(n)$. To get $\mathrm{Spin}(n)$, simply exponentiate these elements and observe that $\left(\frac{1}{2}e_ie_j\right)^2 = -\frac{1}{4}$. We get, by the power series definition,

$$e^{\theta\left(\frac{1}{2}e_ie_j\right)} = \cos(\theta/2) + e_ie_j\sin(\theta/2).$$

As θ goes from 0 to 4π, we get a $U(1)$ subgroup of $\mathrm{Spin}(n)$ generated by $\frac{1}{2}e_ie_j$.

$\mathrm{Spin}(n)$ will act on vectors via

$$v \mapsto \left(e^{\theta\left(\frac{1}{2}e_ie_j\right)}\right) v \left(e^{\theta\left(\frac{1}{2}e_ie_j\right)}\right)^{-1}$$

which rotates the vector v by θ in the ij-plane. Rotating around $\mathrm{Spin}(n)$ once in this plane, i.e. from $\theta = 0$ to $\theta = 4\pi$, we rotate around in $SO(n)$ in the same plane twice. This is because $\mathrm{Spin}(n)$ double covers $SO(n)$.

There is also the concept of Pin groups, which are non-trivial double covers of $O(n)$. They are constructed by considering the other elements of the Clifford algebra.

Of course, we can consider Clifford algebras of non-homogenous signatures. For example, the Lorentz group $SO(3,1)$ is doubly covered by $\mathrm{Spin}(3,1)$, which can be constructed from $C\ell_{3,1}(\mathbb{R})$

Conclusion

We defined and classified finite dimensional real and complex Clifford algebras. The classification for real Clifford algebras has a mod 8 periodicity and the classification for complex Clifford algebras has a mod 2 periodicity. The simple mod 2 periodicity of complex Clifford algebras makes the representation theory of the algebra fairly straightforward to study. We also touched upon the construction of $\mathrm{Spin}(n)$ and motivated why they should be non-trivial double covers of $SO(n)$.

Clifford algebras have interesting applications in a number of fields. It would be interesting to study, for example, the Dirac algebra which is used in quantum field theory from the point of view of it being a Clifford algebra.

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